Divergence-Least Semantics Of amb Is Hoare

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Abstract This note strengthens the hoary observation that McCarthy's `amb` is not monotone with respect to the Smyth and Plotkin powerdomains. It shows that there is no least fixpoint semantics for `amb` that is sensitive to divergence.

This paper is concerned with an *erratic choice* operator $M|M'$, and an *ambiguous choice* operator $M \ \mathtt{amb}\ M'$. Recall that $M|M'$ means: either evaluate M or evaluate M'. And $M \ \mathtt{amb}\ M'$ means: evaluate both M and M' on an arbitrary fair scheduler, and return whatever answer you get first. We defer the study of ambiguous choice until Sect. 2.

1 Erratic Choice

Suppose we have a language $\mathfrak{L}$ containing the following:

– a boolean type `bool`, equipped with constants `t` and `f`, and a conditional operator `if` M `then` N `else` N' at every type
– a natural number type `nat`, equipped with a constant n for each $n \in \mathbb{N}$, and an equality operator $N = N'$
– a term `d` (short for `diverge`) at every type
– an erratic choice operator $|$ at every type

The types `bool` and `nat` are called *ground types.* To describe operational semantics, suppose that we have a function $\mathsf{behs}[-]$

– from the set of closed terms of type `bool` to $\mathcal{P}\{\mathsf{true}, \mathsf{false}, \bot\}$
– from the set of closed terms of type `nat` to $\mathcal{P}(\mathbb{N} \cup \{\bot\})$

satisfying the following equations:

$$\begin{aligned}
\mathsf{behs}[\mathtt{t}] &= \{\mathsf{true}\} \\
\mathsf{behs}[\mathtt{f}] &= \{\mathsf{false}\} \\
\mathsf{behs}[n] &= \{n\} \\
\mathsf{behs}[\mathtt{d}] &= \{\bot\} \\
\mathsf{behs}[M|N] &= \mathsf{behs}[M] \cup \mathsf{behs}[N] \\
\mathsf{behs}[\mathtt{if}\ M\ \mathtt{then}\ N\ \mathtt{else}\ N'] &= \{x \in \mathsf{behs}[N] \mid \mathsf{true} \in \mathsf{behs}[M]\} \\
&\quad \cup \{x \in \mathsf{behs}[N'] \mid \mathsf{false} \in \mathsf{behs}[M']\} \\
&\quad \cup \{\bot \mid \bot \in \mathsf{behs}[M]\}
\end{aligned}$$

$$
\begin{aligned}
\mathsf{behs}[M = N] = &\{\mathsf{true} \mid \exists n \in \mathbb{N}.(n \in \mathsf{behs}[M] \wedge n \in \mathsf{behs}[N])\} \\
&\cup \{\mathsf{false} \mid \exists m, n \in \mathbb{N}.(m \neq n \wedge m \in \mathsf{behs}[M] \wedge n \in \mathsf{behs}[N])\} \\
&\cup \{\bot \mid \bot \in \mathsf{behs}[M] \vee \bot \in \mathsf{behs}[N]\}
\end{aligned}
$$

We write $\mathsf{vals}[N]$ for $\mathsf{behs}[N] \setminus \{\bot\}$, and write $M \Uparrow$ when $\bot \in \mathsf{behs}[M]$. We write $=_{\mathsf{beh}}$ for the kernel of $\mathsf{behs}[-]$.

Some reasonable laws for $\mathfrak{L}$ are shown in Fig. 1, and when we speak of a "denotational semantics", we mean one that validates all these laws. (It is not known whether these laws are complete in any sense.)

Definition 1 If N, N' are of type `bool`, we define $N = N'$ to be

$$
\texttt{if } N \left\{ \begin{array}{ll} \texttt{then} & (\texttt{if } N' \texttt{ then t else f}) \\ \texttt{else} & (\texttt{if } N' \texttt{ then f else t}) \end{array} \right.
$$

□

We call the seven closed terms of type `bool`

$$
\{\texttt{t}, \texttt{f}, \texttt{t}|\texttt{f}, \texttt{d}, \texttt{t}|\texttt{d}, \texttt{f}|\texttt{d}, \texttt{t}|\texttt{f}|\texttt{d}\}
$$

the *basic boolean terms.*

Proposition 1 Let $\lesssim$ be a precongruence on $\mathfrak{L}$ whose symmetrization $\simeq$ satisfies all the laws of Fig. 1. Let $\Gamma \vdash M, M' : B$ be terms.

1. $M|M'$ is $\lesssim$ every upper bound of $\{M, M'\}$, and $\gtrsim$ every lower bound of $\{M, M'\}$.
2. If $M \lesssim M'$ then $M \lesssim M|M' \lesssim M'$.
3. If $M|M'$ is an upper bound of $\{M, M'\}$, then it is a least upper bound.
4. Dually, if $M|M'$ is a lower bound of $\{M, M'\}$, then it is a greatest lower bound.

□

Proof For (1), if P is an upper bound for $\{M, M'\}$, then $M|M' \lesssim P|P \simeq P$. The rest follows. □

Definition 2 We say that a congruence $\simeq$ on $\mathcal{L}$ is *ground-extensional* when $N =_{\mathsf{beh}} N'$ implies $N \simeq N'$ for closed terms N, N' of the same ground type. □

Proposition 2 Let $\lesssim$ be a precongruence on $\mathfrak{L}$ whose symmetrization $\simeq$ satisfies all the laws of Fig. 1.

1. On the basic boolean terms, it takes one of the 20 forms shown in Fig. 2–4.
2. In cases (1), (8), (8^{op}), (4), (11), (11^{op}) we have $M|\texttt{d} \simeq \texttt{d}$ for all $\Gamma \vdash M : B$.
3. In cases (1), (5), (5^{op}), (3), we have $M|\texttt{d} \simeq M$ for all $\Gamma \vdash M : B$.

Laws of Erratic Choice [Plo83]

$$\begin{array}{rcl} M|M' & \simeq & M'|M \\ (M|M')|M'' & \simeq & M|(M'|M'') \\ M|M & \simeq & M \end{array}$$

Laws of Conditionals [Lev04] (Fig. A.8, call-by-name equations)

$$\begin{array}{rcl} \texttt{if t then } M \texttt{ else } M' & \simeq & M \\ \texttt{if f then } M \texttt{ else } M' & \simeq & M' \\ \texttt{if } N \texttt{ then t else f} & \simeq & N \\ \texttt{if (if } N \left\{ \begin{array}{ll} \texttt{then} & N' \\ \texttt{else} & N'' \end{array} \right\} \texttt{) then } M \texttt{ else } M' & \simeq & \texttt{if } N \left\{ \begin{array}{ll} \texttt{then} & (\texttt{if } N' \texttt{ then } M \texttt{ else } M') \\ \texttt{else} & (\texttt{if } N'' \texttt{ then } M \texttt{ else } M') \end{array} \right. \\ \texttt{if d then } M \texttt{ else } M' & \simeq & \texttt{d} \\ \texttt{if } (N|N') \texttt{ then } M \texttt{ else } M' & \simeq & (\texttt{if } N \texttt{ then } M \texttt{ else } M') \\ & & \quad |(\texttt{if } N' \texttt{ then } M \texttt{ else } M') \end{array}$$

Laws of Equality Testing

$$\begin{array}{rcll} c = c & \simeq & \texttt{t} & (c \text{ constant}) \\ c = c' & \simeq & \texttt{f} & (c,\ c' \text{ distinct constants}) \\ \texttt{d} = N & \simeq & \texttt{d} & \\ (N|N') = N'' & \simeq & (N = N'')|(N = N'') & \\ (\texttt{if } N \texttt{ then } N' \texttt{ else } N'') = N''' & \simeq & \texttt{if } N \texttt{ then } (N' = N''') \texttt{ else } (N'' = N''') & \end{array}$$

Laws of Commutativity

$$\begin{array}{rcl} \texttt{if } N \left\{ \begin{array}{ll} \texttt{then} & (\texttt{if } N' \texttt{ then } M \texttt{ else } M') \\ \texttt{else} & (\texttt{if } N' \texttt{ then } M'' \texttt{ else } M''') \end{array} \right. & \simeq & \texttt{if } N' \left\{ \begin{array}{ll} \texttt{then} & (\texttt{if } N \texttt{ then } M \texttt{ else } M'') \\ \texttt{else} & (\texttt{if } N \texttt{ then } M' \texttt{ else } M''') \end{array} \right. \\ N = N' & \simeq & N' = N \end{array}$$

Law of Three Boolean Behaviours

$$(\texttt{if } M \texttt{ then } N \texttt{ else } N')|N|N'|\texttt{d} \ \simeq \ N|N'|\texttt{d}$$

Fig. 1. Laws

4. In cases (1), (5), (5^{op}), (3), (6), (7), (8^{op}), (9^{op}), (11^{op}), (12), we have $M|\mathtt{d} \gtrsim M$ for all $\Gamma \vdash M : B$.
5. Dually, in cases (1), (5), (5^{op}), (3), (6^{op}), (7^{op}), (8), (9), (11), (12^{op}), we have $M|\mathtt{d} \lesssim M$ for all $\Gamma \vdash M : B$.
6. In cases (1),(5),(8), (9), (11), we have $\mathtt{d} \lesssim M$ for all $\Gamma \vdash M : B$.
7. Dually, in cases (1), (5^{op}), (8^{op}), (9^{op}), (11^{op}), we have $\mathtt{d} \gtrsim M$ for all $\Gamma \vdash M : B$.
8. In case (1), we have $M \simeq M'$, for all $\Gamma \vdash M, M' : B$.
9. In cases (1), (5), (6), (8^{op}), the term $M|M'$ is a least upper bound of M and M', for all $\Gamma \vdash M, M' : B$.
10. Dually, in cases (1), (5^{op}), (6^{op}), (8), the term $M|M'$ is a greatest lower bound of M and M', for all $\Gamma \vdash M, M' : B$.
11. In cases (1), (5), (6), (8^{op}), (11^{op}), (12), the term $M|M'|\mathtt{d}$ is a least upper bound of M and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
12. Dually, in cases (1), (5^{op}), (6^{op}), (8), (11), (12^{op}), the term $M|M'|\mathtt{d}$ is a greatest lower bound of M and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
13. In cases (1), (5), (6), (8), (8^{op}), (11^{op}), (12), (4), (9), (10), the term $M|M'|\mathtt{d}$ is a least upper bound of $M|\mathtt{d}$ and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
14. Dually, in cases (1), (5^{op}), (6^{op}), (8), (8^{op}), (11), (12^{op}), (4), (9^{op}), (10^{op}) the term $M|M'|\mathtt{d}$ is a greatest lower bound of $M|\mathtt{d}$ and $M'|\mathtt{d}$ for all $\Gamma \vdash M, M' : B$.
15. Suppose $\simeq$ is ground-extensional. Let N and N' be closed terms of the same ground type. Then $N \lesssim N$ iff

	$N \Uparrow, N' \Uparrow$	$N \Uparrow, N' \not\Uparrow$	$N \not\Uparrow, N' \Uparrow$	$N \not\Uparrow, N' \not\Uparrow$
(1)	true	true	true	true
(2)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(3)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(4)	true	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(5)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$
(5^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(6)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$
(6^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(7)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(7^{op})	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(8)	true	true	false	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(8^{op})	true	false	true	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$
(9)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(9^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(10)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(10^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(11)	true	true	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(11^{op})	true	false	true	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(12)	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(12^{op})	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	$\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$

$\square$

Proof

(1) Exhaustive analysis shows that these are the only preorders on this set for which $|$ and `if` are both monotone.

(2)–(7) Apply `if` $[\cdot]$ `then` M `else` M to the special case where M is `t`.

(8)–(14) We prove these results, using Prop. 1(3)–(4), by applying the context `if` $[\cdot]$ `then` M `else` M' to the special case where M is `t` and M' is `f`.

(15: $\Rightarrow$) We reason as follows.

- Suppose $\mathtt{t}|\mathtt{f} \not\lesssim \mathtt{t}$ and $N \lesssim N'$ and $N \not\Uparrow$, $N' \not\Uparrow$. Then $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$, because $c \in \mathsf{vals}[N] \setminus \mathsf{vals}[N']$ would imply

$$\begin{aligned}\mathtt{t}|\mathtt{f} &=_{\mathsf{beh}} (\mathtt{if}\ (N = c)\ \mathtt{then\ f\ else\ t})|\mathtt{t} \\ &\lesssim\ (\mathtt{if}\ (N' = c)\ \mathtt{then\ f\ else\ t})|\mathtt{t} =_{\mathsf{beh}} \mathtt{t}\end{aligned}$$

  Dually, if $\mathtt{t}|\mathtt{f} \not\gtrsim \mathtt{t}$ and $N \lesssim N'$ and $N \not\Uparrow, N' \not\Uparrow$, then $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$.
- Suppose $\mathtt{t}|\mathtt{d} \not\lesssim \mathtt{t}$ and $N \lesssim N'$ and $N' \not\Uparrow$. Then $N \not\Uparrow$, because $N \Uparrow$ would imply

$$\begin{aligned}\mathtt{t}|\mathtt{d} &=_{\mathsf{beh}} (\mathtt{if}\ (N = N)\ \mathtt{then\ t\ else\ t})|\mathtt{t} \\ &\lesssim\ (\mathtt{if}\ (N' = N')\ \mathtt{then\ t\ else\ t})|\mathtt{t} =_{\mathsf{beh}} \mathtt{t}\end{aligned}$$

  Dually, if $\mathtt{t}|\mathtt{d} \not\gtrsim \mathtt{t}$ and $N \lesssim N'$ and $N \not\Uparrow$, then $N' \not\Uparrow$.
- Suppose $\mathtt{t}|\mathtt{f}|\mathtt{d} \not\lesssim \mathtt{t}|\mathtt{d}$, and $N \lesssim N'$. Then $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$, because $c \in \mathsf{vals}[N] \setminus \mathsf{vals}[N']$ would imply

$$\begin{aligned}\mathtt{t}|\mathtt{f}|\mathtt{d} &=_{\mathsf{beh}} (\mathtt{if}\ N = c\ \mathtt{then\ f}|\mathtt{t})|\mathtt{t}|\mathtt{d} \\ &\lesssim\ (\mathtt{if}\ N' = c\ \mathtt{then\ f}|\mathtt{t})|\mathtt{t}|\mathtt{d} =_{\mathsf{beh}} \mathtt{f}|\mathtt{d}\end{aligned}$$

  Dually, if $\mathtt{t}|\mathtt{f}|\mathtt{d} \not\gtrsim \mathtt{t}|\mathtt{d}$ and $N \lesssim N'$, then $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$.

(15: $\Leftarrow$) We reason as follows. Suppose $N \Uparrow, N' \Uparrow$.

- In the cases where Prop. 2(2) holds, we have

$$N =_{\mathsf{beh}} N|\mathtt{d} \simeq \mathtt{d} \simeq N'|\mathtt{d} =_{\mathsf{beh}} N'$$

- In the cases where Prop. 2(13) holds, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N|\mathtt{d} \lesssim N|N'|\mathtt{d} =_{\mathsf{beh}} N'$$

  Dually, if $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$, then, in the cases where Prop. 2(14) holds, we have $N \lesssim N'$.
- If $\mathsf{vals}[N] = \mathsf{vals}[N']$, then $N =_{\mathsf{beh}} N'$ so $N \lesssim N'$.

Suppose $N \Uparrow, N' \not\Uparrow$.

- In case (1), by Prop. 2(8), we have $N \lesssim N'$.
- In the cases where Prop. 2(13) and Prop. 2(5) both hold, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N|\mathtt{d} \lesssim N|N'|\mathtt{d} =_{\mathsf{beh}} N'|\mathtt{d} \lesssim N'$$

- In the cases where Prop. 2(12) holds, $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N|N'|\mathtt{d} \lesssim N'$$

- In the cases where Prop. 2(5) holds, $\mathsf{vals}[N] = \mathsf{vals}[N']$ implies

$$N =_{\mathsf{beh}} N'|\mathtt{d} \lesssim N'$$

Dually, suppose $N \not\Uparrow, N' \Uparrow$.

- In case (1), we have $N \lesssim N'$.
- In the cases where Prop. 2(14) and Prop. 2(4) both hold, $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$ implies $N \lesssim N'$
- In the cases where Prop. 2(11) holds, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies $N \lesssim N'$.
- In the cases where Prop. 2(4) holds, $\mathsf{vals}[N] = \mathsf{vals}[N']$ implies $N \lesssim N'$.

Suppose $N \not\Uparrow, N' \not\Uparrow$.

- In case (1), by Prop. 2(8), we have $N \lesssim N'$.
- In cases where Prop. 2(9) holds, $\mathsf{vals}[N] \subseteq \mathsf{vals}[N']$ implies

$$N \lesssim N|N' =_{\mathsf{beh}} N'$$

- Dually, in cases where Prop. 2(10) holds, $\mathsf{vals}[N] \supseteq \mathsf{vals}[N']$ implies $N \lesssim N'$.
- If $\mathsf{vals}[N] = \mathsf{vals}[N']$ then $N =_{\mathsf{beh}} N'$ so $N \lesssim N'$.

□

In the cases where Prop. 2(6) applies, we say that $\lesssim$ is *divergence-least.*

Since any congruence is a precongruence, we can specialize Prop. 2 as follows.

Proposition 3 Let $\simeq$ be a congruence on $\mathfrak{L}$ satisfying the laws of Fig. 1.

1. On the basic boolean terms, it takes one of the forms (1), (2), (3), (4).
2. In cases (1), (4), we have $M|\mathtt{d} \simeq \mathtt{d}$ for all $\Gamma \vdash M : B$.
3. In cases (1), (3), we have $M|\mathtt{d} \simeq M$ for all $\Gamma \vdash M : B$.
4. In case (1), we have $M \simeq M'$, for all $\Gamma \vdash M, M' : B$.
5. Suppose $\simeq$ is ground-extensional, and let N and N' be closed terms of the same ground type. Then $N \simeq N'$ iff

	$N \Uparrow, N' \Uparrow$	$N \Uparrow, N' \not\Uparrow$	$N \not\Uparrow, N' \Uparrow$	$N \not\Uparrow, N' \not\Uparrow$
(1)	true	true	true	true
(2)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(3)	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$	$\mathsf{vals}[N] = \mathsf{vals}[N']$
(4)	true	false	false	$\mathsf{vals}[N] = \mathsf{vals}[N']$

□

In the cases where Prop. 3(3) applies, we say that $\simeq$ is *divergence-insensitive.*

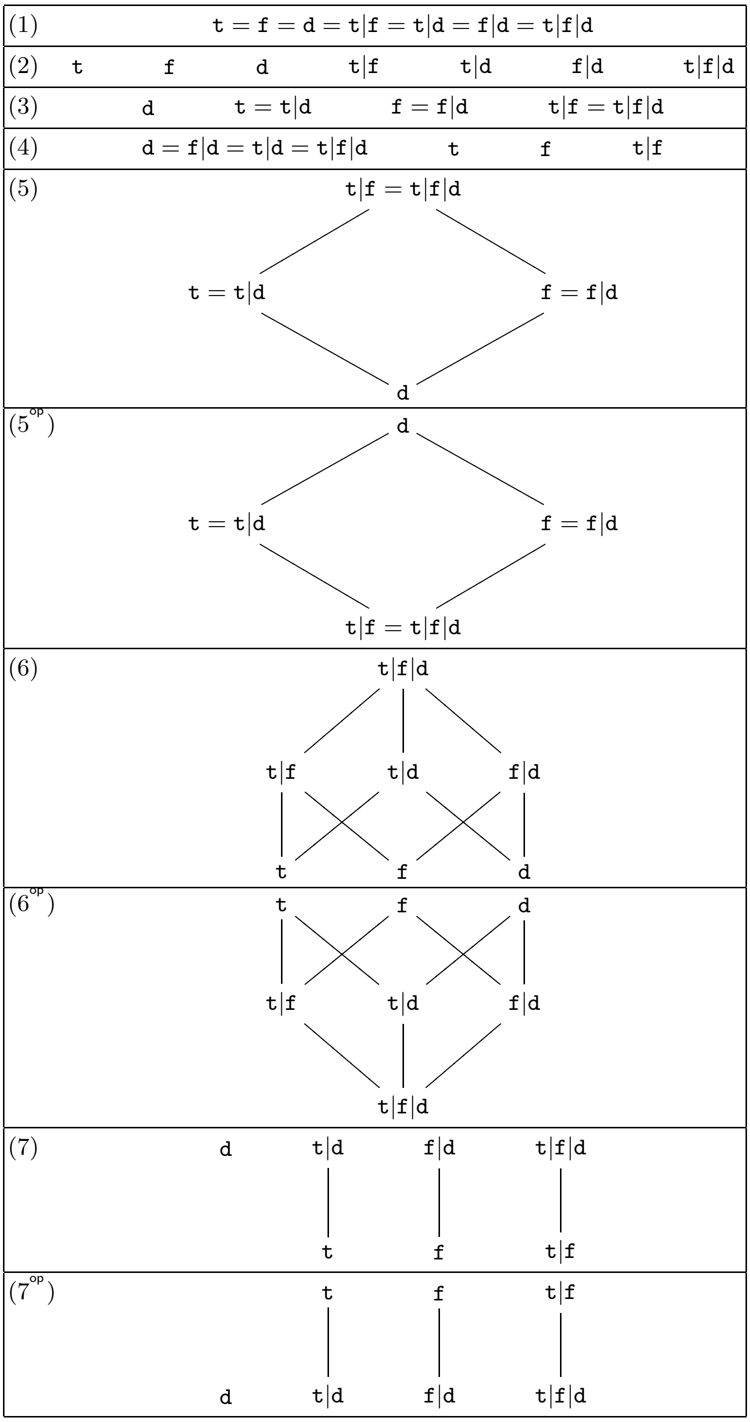

Fig. 2. The Twenty Precongruences

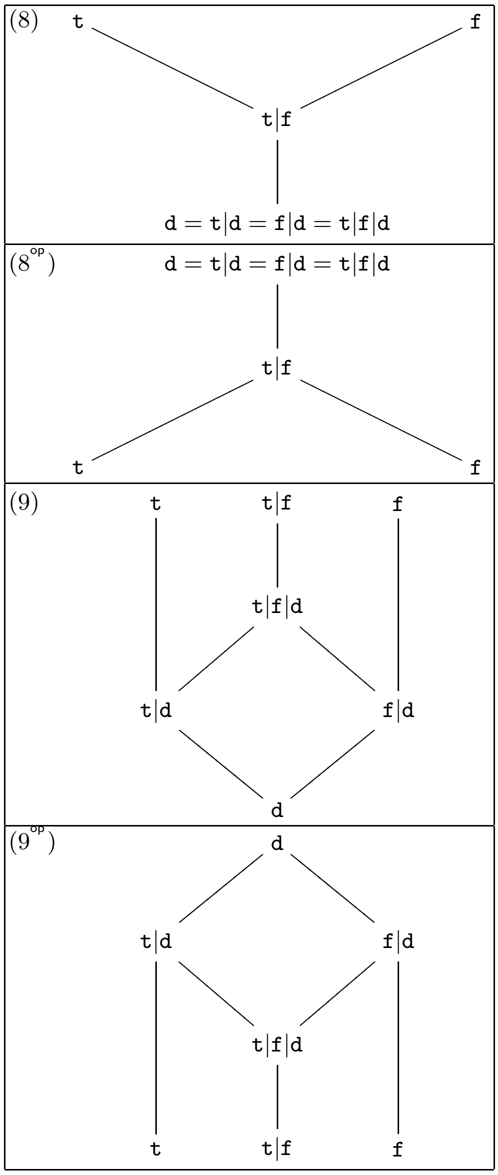

Fig. 3. The Twenty Precongruences (continued)

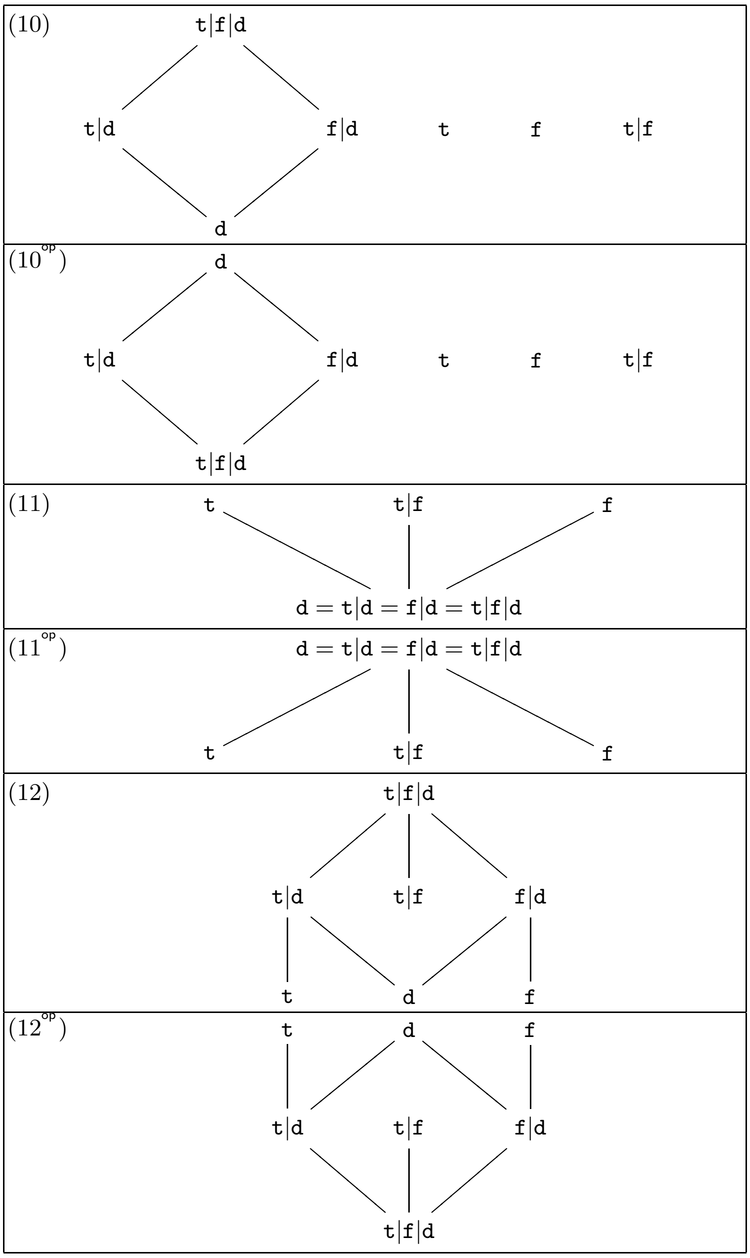

Fig. 4. The Twenty Precongruences (continued)

2 Ambiguous Choice

Suppose that $\mathfrak{L}$ contains an ambiguous choice operator amb—not necessarily at every type, but at least at type bool, and the function behs[−] has the property

$$\begin{aligned}\mathsf{behs}[M \;\mathtt{amb}\; N] = ((&\mathsf{behs}[M] \cup \mathsf{behs}[N]) \setminus \bot) \\ &\cup \{\bot \mid \bot \in \mathsf{behs}[M] \wedge \bot \in \mathsf{behs}[N]\}\end{aligned}$$

Laws pertaining to this operator are shown in Fig. 5. We can deduce from them the equation

$$(M|\mathtt{d}) \;\mathtt{amb}\; (N|\mathtt{d}) \simeq M|N|\mathtt{d} \tag{1}$$

as follows. The RHS $\simeq$

$$(M|N|\mathtt{d}) \;\mathtt{amb}\; (M|N|\mathtt{d}) \tag{2}$$

We expand both (2) and the LHS of (1) by distributing amb over |, and in each case we obtain

$$(M \;\mathtt{amb}\; N)|M|N|\mathtt{d}$$

All the laws of Fig. 1 and 5 are satisfied by the congruence in [LM99] if the language treated there is extended with cost-free conditionals. All but the "laws of commutativity" are satisfied by the congruence in [Las05].

$$\begin{array}{rcll}
N \;\mathtt{amb}\; N' & \simeq & N' \;\mathtt{amb}\; N & \\
(N \;\mathtt{amb}\; N') \;\mathtt{amb}\; N'' & \simeq & N \;\mathtt{amb}\; (N' \;\mathtt{amb}\; N'') & \\
N \;\mathtt{amb}\; N & \simeq & N & \\
c \;\mathtt{amb}\; c' & \simeq & c|c' & (c,\, c' \text{ constants}) \\
\mathtt{d} \;\mathtt{amb}\; N & \simeq & N & \\
(N|N') \;\mathtt{amb}\; N'' & \simeq & (N \;\mathtt{amb}\; N'')|(N' \;\mathtt{amb}\; N'') & \\
(N \;\mathtt{amb}\; N')|N'' & \simeq & (N|N'') \;\mathtt{amb}\; (N'|N'') &
\end{array}$$

Fig. 5. Laws of Ambiguous Choice

Proposition 4 Let $\simeq$ be a congruence on $\mathfrak{L}$ satisfying all the laws of Fig. 1–5. Then $\simeq$ is divergence-insensitive iff

$$N|N' \simeq N \;\mathtt{amb}\; N' \tag{3}$$

for all $\Gamma \vdash N, N' : B$ where B is an amb type. □

Proof If $\simeq$ is divergence-insensitive, then

$$\begin{aligned}M \;\mathtt{amb}\; N &\simeq (M|\mathtt{d}) \;\mathtt{amb}\; (N|\mathtt{d}) \\ &\simeq M|N|\mathtt{d} \\ &\simeq M|N\end{aligned}$$

Conversely, (3) implies $\mathtt{t}|\mathtt{d} \simeq \mathtt{t} \;\mathtt{amb}\; \mathtt{d} \simeq \mathtt{t}$. □

Proposition 5 Any divergence-insensitive denotational semantics of the `amb`-free fragment of $\mathcal{L}$ has a unique extension to a denotational semantics of $\mathcal{L}$. It is obtained by setting $[\![N \texttt{ amb } N']\!]$ to be $[\![N|N']\!]$. $\square$

Proof It is trivial to check the laws for ambiguous choice. Uniqueness follows from Prop. 4. $\square$

Proposition 6 1. Let $\lesssim$ be a precongruence on $\mathfrak{L}$ whose symmetrization $\lesssim$ satisfies all the laws of Fig. 1 and Fig. 5. On the basic boolean terms, it takes one of the forms (1), (2), (3), (5), (5^{op}), (6), (6^{op}), (7), (7^{op}). Hence if $\lesssim$ is divergence-least, then it is divergence-insensitive.

2. Let $\simeq$ be a congruence on $\mathfrak{L}$ satisfying all the laws of Fig. 1 and 5. On the basic boolean terms, it takes one of the forms (1), (2), (3).

$\square$

Proof These are the only cases for which `amb` is monotone. $\square$

If $\mathcal{L}$ contains recursion, then, for any semantics that interprets recursion as a least fixpoint, the induced precongruence will be divergence-least. In a call-by-name language, for example, `diverge` can be expressed as $\mu\texttt{x}.\texttt{x}$, so it denotes the least fixpoint of the identity function. Therefore, Prop. 6(1) shows that there cannot be a least fixpoint semantics that is divergence-sensitive.

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