BOOLEAN PRECONGRUENCES

PAUL BLAIN LEVY

University of Birmingham, Birmingham B15 2TT, UK
e-mail address: pbl@cs.bham.ac.uk

Abstract. We investigate the possible ways of ordering terms of ground type in a non-deterministic (or deterministic) language that contains erroneous behaviours such as divergence, crash or deadlock. We see that the ordering at boolean type, called a "boolean precongruence", is key: it determines the ordering at other ground types, and induces a contextual preorder. We examine the circumstances in which amb is monotone, and in which the ordering at Sierpinski type or even zero type suffices.

Each boolean precongruence gives a way of lifting relations, leading to a power-poset construction. We obtain a notion of simulation, and give general conditions for when a modal logic is sound and complete for the induced similarity preorder.

Contents

1998 ACM Subject Classification: F.3.2.

Key words and phrases: nondeterminism, amb, divergence, boolean, order, precongruence, simulation, modal logic, bisimulation.

Supported by ESPRC Advanced Research Fellowship EP/E056091/1.

****
****

****

© Paul Blain Levy
Creative Commons

1. Introduction

A familiar scenario in semantics research is that we have a typed language $\mathcal{L}$, with binary nondeterminism (`or`) and divergent terms.

Certain types are designated *ground types*, each equipped with a set of *values*. For example we might have a boolean type, a natural number type, a Sierpinski type (with just one value) and a zero type (with no values). The language provides a `case` operation for each finite ground type, and a `case`...`else` construction for each infinite ground type.

We then consider various precongruences on $\mathcal{L}$. The first question that arises is: how should we order the terms of a ground type? For ground type, three choices are traditionally considered: the lower, upper and convex orderings. They give rise to the powerdomain constructions associated with Hoare, Smyth and Plotkin respectively [Plo76, Smy78], and hence to a least fixpoint semantics of recursion. In particular divergence, recursively defined to be itself, is a least element.

Are these the only possible orderings at ground type? The goal of this paper is to consider this question, and its ramifications, in a systematic manner.

We begin with the boolean type. We surely want `if` to be monotone, which implies monotonicity of `or` because of the equation

$$M \texttt{ or } N = \texttt{if } (\texttt{true or false}) \texttt{ then } M \texttt{ else } N \tag{1.1}$$

So we list all the "boolean precongruences" i.e. all possible orderings that have these properties. Of course, there can only be finitely many. We examine which ones make divergence a least element: surprisingly, there is a fourth, which we call "smash".

One might expect to have to perform a similar listing at other ground types. But that is unnecessary: the ordering at the boolean type determines the ordering at all other ground types.

We analyze some variants of the above situation. One is where there is a finite set E of "errors" that might include divergence, crash, deadlock etc. This highlights the structure of many definitions and results, which easily admit this generalization.

Other variants are where the language is deterministic, or provides McCarthy's amb. The non-monotonicity of amb wrt certain orderings is notorious, so—continuing the work of [LLP05]—we look at which orders are acceptable.

Each boolean precongruence gives rise to a contextual preorder, and in Sect. 3.1 we see some instances of these preorders that have appeared in the literature. Although our work demonstrates that contexts of boolean type always suffice, we examine in Sect. 3.3 when it is acceptable to restict attention further, to contexts of Sierpinski type or even zero type. This analysis exploits some results, presented in Sect. 3.2, saying that certain boolean precongruences can be represented as intersections of canonical ones.

In Sect. 4, we see how each boolean precongruence gives a way of lifting relations (Sect. 4.1), and these leads in two directions.

- to a free construction (Sect. 4.2), generalizing the definition of our orders at ground types in the form of a "power-poset".

- a notion of simulation for transition systems (Sect. 4.3), equipped with a modal logic (Sect. 4.4).

Each of these developments is familiar from the literature; the contribution of this paper is simply a more systematic analysis. In particular, various notions of simulation have been proposed for transition systems with divergence [Abr91, AH92, HP80, How96, Mil81, Ong93, vG01, vG93, Wal90], and some have been explicitly linked to orderings such as lower and upper [Las98, Mor98, Pit01, Uli92]. Our account explains the connection.

A theme of the paper is that requirements can be reduced to simple conditions on booleans that can be mechanically checked. In particular:

- an ordering on ground type is determined by an ordering at boolean type (Prop. 2.19)
- compatibility of a semilattice with a power-poset construction (Prop. 4.7) is determined by some boolean conditions (Def. 2.12)
- the monotonicity of amb at all ground types (Prop. 2.23) is determined by its monotonicity at boolean type (Def. 2.21(2))
- the sufficiency of Sierpinski type or zero type contexts is easily checked (Sect. 3.2)
- the soundness and completeness of modal logics for similarities (Prop. 4.14 and Prop. 4.16) is determined by some boolean properties of the modalities (Def. 4.13 and Def. 4.15).

Notation

$$\begin{array}{rcl}
\mathcal{P}^{>0}A & \stackrel{\mathrm{def}}{=} & \{B \subseteq A \mid B \text{ nonempty}\} \\
\mathcal{P}^{>0,<\aleph_0}A & \stackrel{\mathrm{def}}{=} & \{B \subseteq A \mid B \text{ nonempty and finite}\} \\
\mathcal{P}^{>0,\leqslant\aleph_0}A & \stackrel{\mathrm{def}}{=} & \{B \subseteq A \mid B \text{ nonempty and countable}\} \\
\mathbb{B} & = & \{\mathsf{t}, \mathsf{f}\} \qquad \text{(the set of booleans)}
\end{array}$$

2. Ordering Terms

2.1. Ordering Terms of Boolean Type.

Let us write out explicitly the semantic form of the `case` construction. (The prefixes "D" and "ND" stand for "deterministic" and "nondeterministic".)

Definition 2.1.

(1) Let E be a set. For sets X and Y, we write $\mathsf{Dcase}^E_{X,Y}$ for the function

$$\begin{array}{ccccc}
X+E & \times & (Y+E)^X & \longrightarrow & Y+E \\
\mathsf{inl}\ b, & & f & \mapsto & f(b) \\
\mathsf{inr}\ e, & & f & \mapsto & \mathsf{inr}\ e
\end{array}$$

For $X = \mathbb{B}$, we call it $\mathsf{Dif}^E_Y \ :\ (\mathbb{B}+E) \times (Y+E)^{\mathbb{B}} \longrightarrow Y+E$.

(2) Let E be a set. For sets X and Y, we write $\mathsf{NDcase}^E_{X,Y}$ for the function

$$\begin{array}{ccccc}
\mathcal{P}(X+E) & \times & (\mathcal{P}(Y+E))^X & \longrightarrow & \mathcal{P}(Y+E) \\
K+D, & & f & \mapsto & \bigcup_{b \in K} f(b) \cup \{\mathsf{inr}\ e \mid e \in D\}
\end{array}$$

In the case $X = \mathbb{B}$, we write

$$\mathsf{NDif}^E_Y \ :\ \mathcal{P}^{>0}(\mathbb{B}+E) \times (\mathcal{P}^{>0}(Y+E))^{\mathbb{B}} \longrightarrow \mathcal{P}^{>0}(Y+E)$$

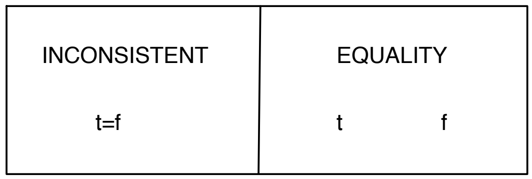

Figure 1: All the DBPs for no errors

for the restriction to nonempty sets.

Remark 2.2. For any monad T on **Set**, and sets X and Y, the *Kleisli extension* map $\mathsf{kl}^T_{X,Y}$ is given by

$$\begin{array}{ccccc} TX & \times & (TY)^X & \longrightarrow & TY \\ x, & & f & \mapsto & (\mu Y)((Tf)x) \end{array}$$

Def. 2.1 can be regarded as consisting of special cases of this:

- $\mathsf{Dcase}^E_{X,Y}$ is $\mathsf{kl}^T_{X,Y}$ where $T : Z \mapsto Z + E$
- $\mathsf{Dif}^E_X Y$ is $\mathsf{kl}^T_{\mathbb{B},Y}$ where $T : Z \mapsto Z + E$
- $\mathsf{NDcase}^E_{X,Y}$ is $\mathsf{kl}^T_{X,Y}$ where $T : Z \mapsto \mathcal{P}(Z + E)$
- $\mathsf{NDif}^E_X Y$ is $\mathsf{kl}^T_{\mathbb{B},Y}$ where $T : Z \mapsto \mathcal{P}^{>0}(Z + E)$.

We shall not need this more abstract formulation.

The key definition of the paper is the following.

Definition 2.3.

(1) Let E be a set. A *deterministic boolean precongruence* (DBP) for E-errors is a preorder on $\mathbb{B} + E$ making $\mathsf{Dif}^E_{\mathbb{B}}$ monotone. If symmetric, it is a *deterministic boolean congruence* (DBC) for E-errors.

(2) Let E be a finite set. A *nondeterministic boolean precongruence* (NDBP) for E-errors is a preorder on $\mathcal{P}^{>0}(\mathbb{B} + E)$ making $\mathsf{NDif}^E_{\mathbb{B}}$ and $\cup$ monotone. If symmetric, it is a *nondeterministic boolean congruence* (NDBC) for E-errors.

Clearly any precongruence of interest on our language $\mathcal{L}$ provides a NDBP (or DBP).

Remark 2.4. The exclusion of the empty set in Def. 2.3(2) is immaterial. We shall see below (Prop. 2.16(2d)) that every NDBP for E-errors extends uniquely to a preorder on $\mathcal{P}(\mathbb{B} + E)$ making $\mathsf{NDcase}^E_{\mathbb{B},\mathbb{B}}$ and $\cup$ monotone.

Remark 2.5. The requirement for $\cup$ to be monotone in Def. 2.3(2) is redundant, because of (1.1), which takes the form

$$x \cup y = \mathsf{NDif}^E_{\mathbb{B}}(\{\mathsf{inl}\ \mathsf{t}, \mathsf{inl}\ \mathsf{f}\}, \{\mathsf{t}.x, \mathsf{f}.y\})$$

Remark 2.6. If E is a *countable* set, then we can still use Def. 2.3(2) provided we additionally require countable union to be monotone. (This is automatic if E is finite.) The results in the paper all remain valid, but see Remarks 2.22 and 4.8.

The boolean precongruences in some cases of interest are listed in Fig. 1–5.

The DBPs for divergence and crash are all intersections of BIPOINTED and DOUBLE POINTED, and their converses. The names “stable”, “costable”, “bistable” and “bistable coherence” are taken from [Lai07a, Lai05].

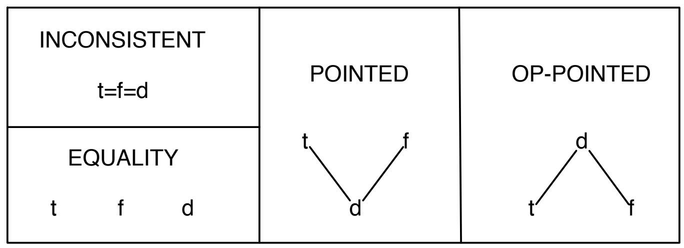

Figure 2: All the DBPs for divergence

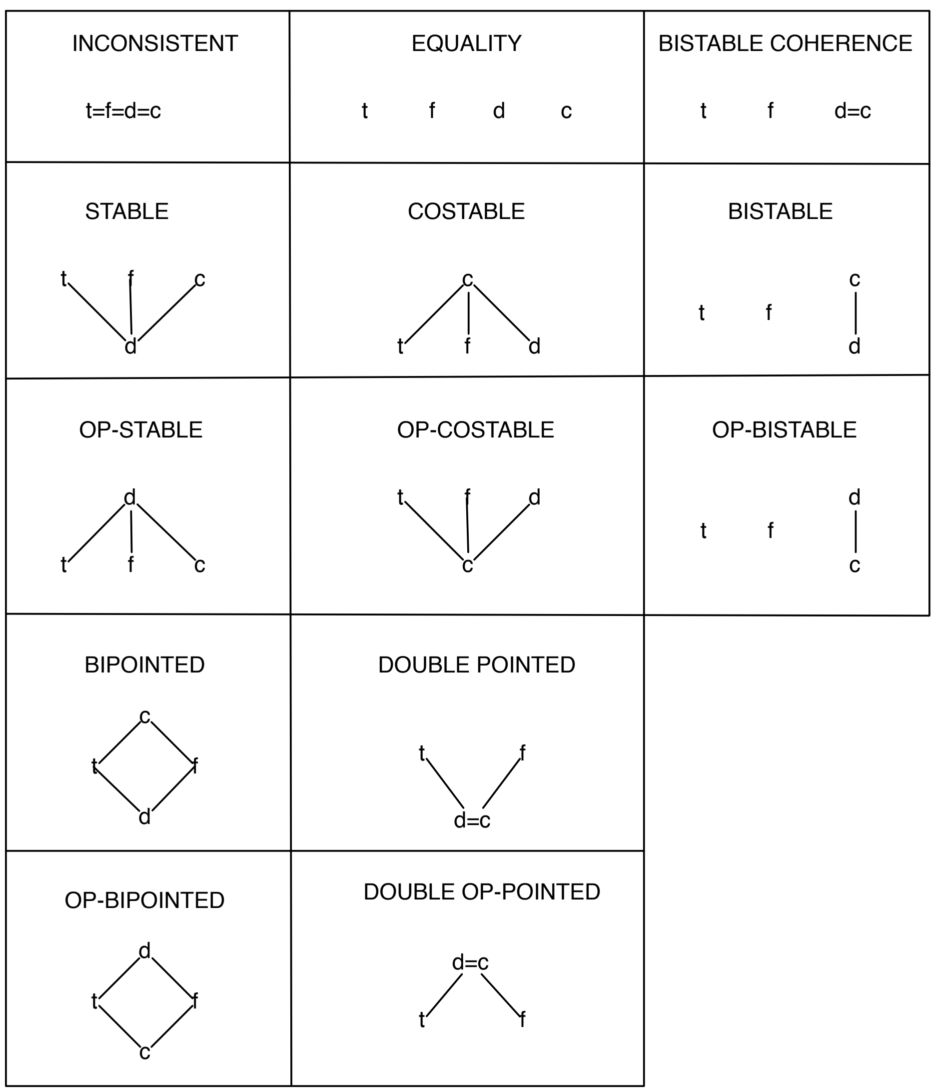

Figure 3: All the DBPs for divergence and crash

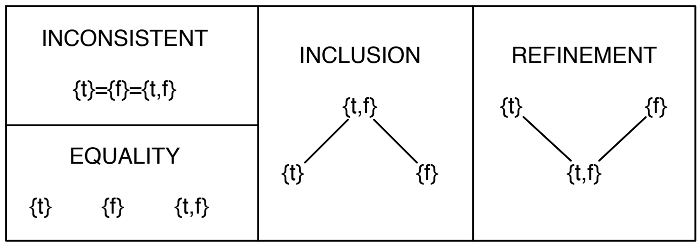

Figure 4: All the NDBPs for no errors

The NDBPs for divergence are all intersections of LOWER, UPPER and SMASH, and their converses. Although SMASH is stronger than UPPER, they have the symmetrization (viz. UPPER CONGRUENCE) and the same intersection with LOWER (viz. CONVEX).

We do not list the NDBPs for divergence and crash, as they are numerous. But in [Lai06, Lai07b, Lai09], five NDBPs are studied, displayed in Fig. 6. (Warning: in [Lai06], both the may and the costable preorders are reversed.)

The following is repeatedly useful.

Lemma 2.7. *Let $\sqsubseteq$ be a NDBP for a finite set E. Then for any $D, D' \subseteq E$, we have*

$$\begin{array}{lrcl} & \{\mathsf{t}\} + D & \sqsubseteq & \{\mathsf{f}\} + D' \\ \textit{iff both} & \{\mathsf{t}\} + D & \sqsubseteq & \{\mathsf{t},\mathsf{f}\} + D' \\ \textit{and} & \{\mathsf{t},\mathsf{f}\} + D & \sqsubseteq & \{\mathsf{f}\} + D' \end{array}$$

Proof. ($\Rightarrow$) is obvious, and ($\Leftarrow$) is given by

$$\begin{array}{rcl} \{\mathsf{t}\} + D & = & (\{\mathsf{t}\} + D) \cup (\{\mathsf{t}\} + D) \\ & \sqsubseteq & (\{\mathsf{t}\} + D) \cup (\{\mathsf{t},\mathsf{f}\} + D') \\ & = & (\{\mathsf{t},\mathsf{f}\} + D) \cup (\{\mathsf{f}\} + D') \\ & \sqsubseteq & (\{\mathsf{f}\} + D') \cup (\{\mathsf{f}\} + D') \\ & = & \{\mathsf{f}\} + D' \end{array}$$

$\square$

There is one DBP for E-errors in which

$$\mathsf{inl\ true} \sqsubseteq \mathsf{inl\ false}$$

and one NDBP for E-errors in which

$$\{\mathsf{inl\ true}\} \sqsubseteq \{\mathsf{inl\ false}\}$$

Both are called INCONSISTENT. Any other boolean precongruence is said to be *consistent*.

Remark 2.8. A deterministic calculus with divergence and deadlock is studied in [BL95], but it does not include sequencing. For this reason, the "vertical" ordering presented there is not a DBP, although the "standard" and "flat" orderings correspond to DOUBLE POINTED and STABLE respectively.

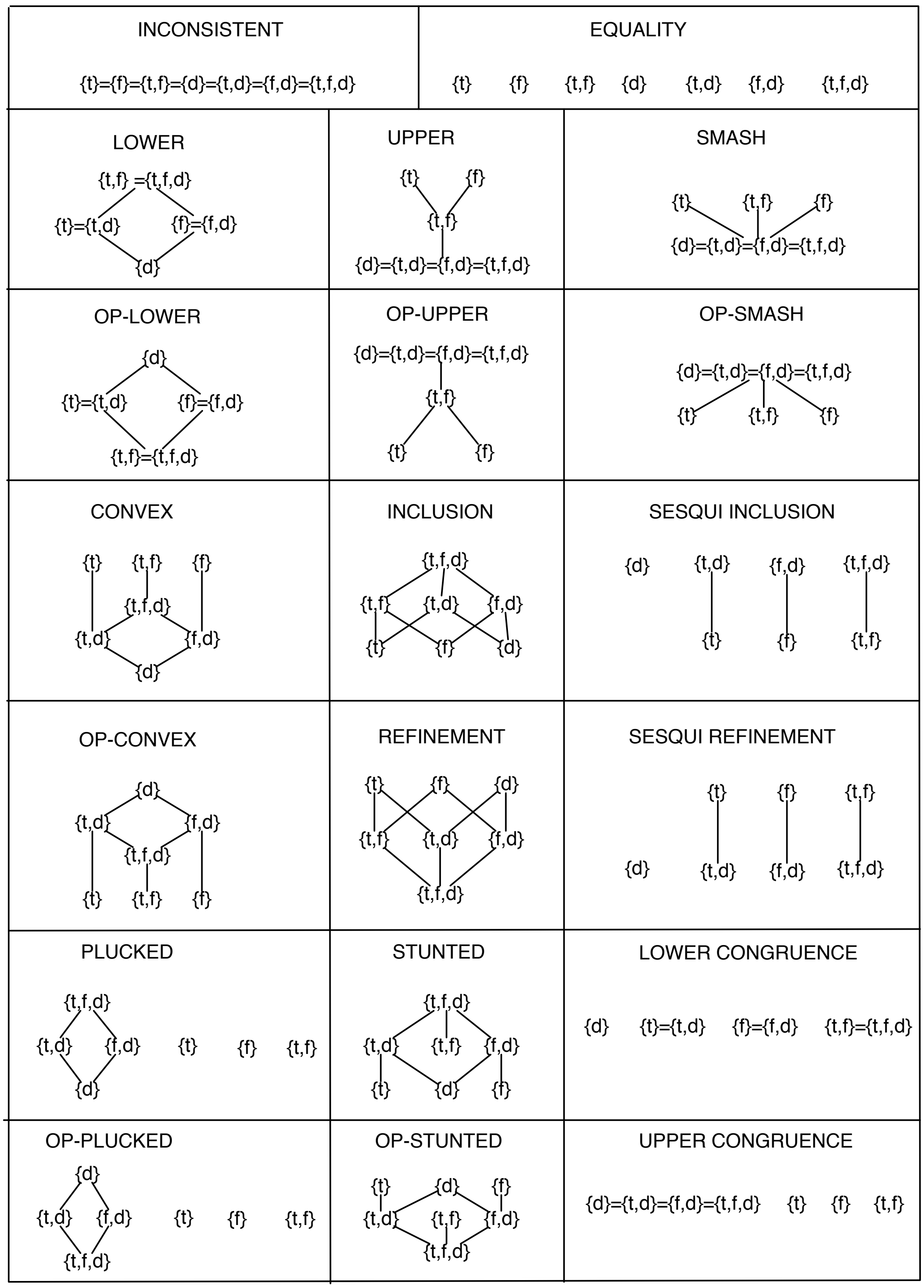

Figure 5: All the NDBPs for divergence

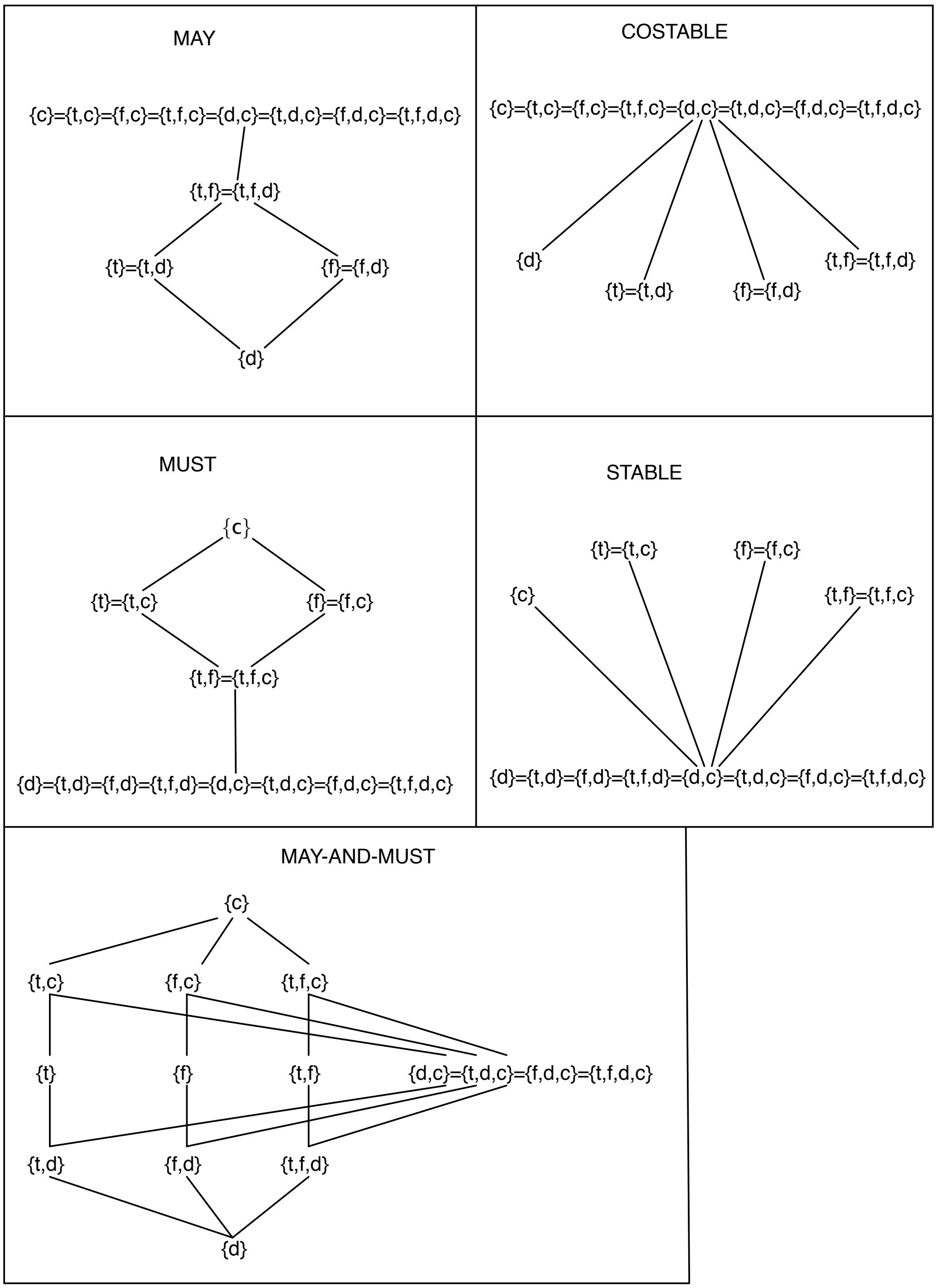

Figure 6: NDBPs for divergence and crash described in [Lai06, Lai07b, Lai09]

2.2. **Preordered Semilattices.** In this section, we examine the algebraic structure of the set of terms of any type.

Definition 2.9. (1) A *semilattice* is a set X equipped with a binary operation or satisfying

$$\begin{array}{llrcl} \textbf{Commutativity} & \forall x, y \in X. & x \mathbin{\mathsf{or}} y & = & y \mathbin{\mathsf{or}} x \\ \textbf{Associativity} & \forall x, y, z \in X. & (x \mathbin{\mathsf{or}} y) \mathbin{\mathsf{or}} z & = & x \mathbin{\mathsf{or}} (y \mathbin{\mathsf{or}} x) \\ \textbf{Idempotency} & \forall x \in X. & x \mathbin{\mathsf{or}} x & = & x \end{array}$$

(2) Let $\underline{B} = (X, \mathsf{or})$ be a semilattice. An I-ary operation $X^I \xrightarrow{\mathsf{choose}} X$ is a *join operation* for $\underline{B}$ when it satisfies
- for $I = \emptyset$

$$\begin{array}{llrcl} \textbf{Neutrality} & \forall x \in X. & x \mathbin{\mathsf{or}} \mathsf{choose}\{\} & = & x \end{array}$$

- for $I \neq \emptyset$

$$\begin{array}{llrcl} \textbf{Absorption} & \forall \hat{\imath} \in I, x \in X^I. & x_{\hat{\imath}} \mathbin{\mathsf{or}} \mathsf{choose}_{i \in I}\ x_i & = & \mathsf{choose}_{i \in I}\ x_i \\ \textbf{Idempotency} & \forall x \in X. & \mathsf{choose}_{i \in I}\ x & = & x \\ \textbf{Distributivity} & \forall x \in X, y \in X^I. & x \mathbin{\mathsf{or}} \mathsf{choose}_{i \in I}\ y_i & = & \mathsf{choose}_{i \in I}\ (x \mathbin{\mathsf{or}} y_i) \end{array}$$

We recall that a semilattice operation or on X corresponds to a partial order wrt which every two elements have a least upper bound $x \mathbin{\mathsf{or}} y$. Explicitly the order is given by

$$\leqslant_{\mathsf{or}} \stackrel{\text{def}}{=} \{(x, y) \in X^2 \mid x \mathbin{\mathsf{or}} y = y\}$$

Moreover, an I-ary operation choose is a join operation iff $\mathsf{choose}_{i \in I} x_i$ is the least upper bound of $\{x_i\}_{i \in I}$ wrt $\leqslant_{\mathsf{or}}$. Consequently an I-ary join operation is unique when it exists.

Definition 2.10.

(1) Let E be a set. Let X be a set and let $\underline{Y}$ be a *set with E-errors* i.e. a set Y equipped with a function $E \xrightarrow{\mathsf{raise}} Y$. We write $\mathsf{Dcase}^E_{X,\underline{Y}}$ for the function

$$\begin{array}{cccc} (X + E) & \times & Y^X & \longrightarrow & Y \\ \mathsf{inl}\ b, & & f & \mapsto & f(b) \\ \mathsf{inr}\ e, & & f & \mapsto & \mathsf{raise}\ e \end{array}$$

For $X = \mathbb{B}$, we call it $\mathsf{Dif}^E_{\underline{Y}} : (\mathbb{B} + E) \times Y^{\mathbb{B}} \longrightarrow Y$.

(2) Let E be a finite set. Let X be a set and let $\underline{Y}$ be a *semilattice with E-errors* i.e. a semilattice (Y, or) equipped with a function $E \xrightarrow{\mathsf{raise}} Y$. We write $\mathsf{NDcase}^E_{X,\underline{Y}}$ for the partial function

$$\begin{array}{cccccl} \mathcal{P}(X + E) & \times & Y^A & \longrightarrow & Y & \\ x, & & f & \mapsto & \mathsf{choose}_x \begin{cases} \mathsf{inl}\ a. & f a \\ \mathsf{inr}\ e. & \mathsf{raise}\ e \end{cases} & \\ & & & & \text{if } (Y, \mathsf{or}) \text{ has an } x\text{-ary join operation} & \\ & & & \mapsto & \text{undefined otherwise} & \end{array}$$

For $X = \mathbb{B}$, it restricts to a total function

$$\mathsf{NDif}^E_{\underline{Y}} : \mathcal{P}^{>0}(\mathbb{B} + E) \times Y^{\mathbb{B}} \longrightarrow Y$$

.

Remark 2.11. For any monad T on **Set**, set X and T-algebra $\underline{B} = (Y, \theta)$, the *Kleisli extension* map $\mathsf{kl}^T_{X,\underline{B}}$ is given by

$$\begin{array}{ccccc} TX & \times & Y^X & \longrightarrow & Y \\ x, & & f & \mapsto & (\theta)((Tf)x) \end{array}$$

Def. 2.10 can be regarded as consisting of special cases of this:

- $\mathsf{Dcase}^E_{X,\underline{B}}$ is $\mathsf{kl}^T_{X,\underline{B}}$ where $T : Z \mapsto Z + E$
- $\mathsf{Dif}^E_X \underline{B}$ is $\mathsf{kl}^T_{\mathbb{B},\underline{B}}$ where $T : Z \mapsto Z + E$
- $\mathsf{NDcase}^E_{X,\underline{B}}$ is $\mathsf{kl}^T_{X,\underline{B}}$ where T is any submonad of $Z \mapsto \mathcal{P}(Z + E)$ for which $\underline{B}$ is an algebra
- $\mathsf{NDif}^E_X \underline{B}$ is $\mathsf{kl}^T_{\mathbb{B},\underline{B}}$ where $T : Z \mapsto \mathcal{P}^{>0}(Z + E)$.

We shall not need this more abstract formulation.

Definition 2.12.

(1) Let E be a set.
- A *preordered set with E-errors* $\underline{B}$ is a preordered set $(X, \leqslant)$ equipped with a function $E \xrightarrow{\mathsf{raise}} X$.
- Such a $\underline{B}$ is *compatible* with a DBP for E-errors $\sqsubseteq$ when
  (a) $\mathsf{inl}\ \mathsf{t} \sqsubseteq \mathsf{inl}\ \mathsf{f}$ implies $x \leqslant y$ for all $x, y \in B$
  (b) $\mathsf{inr}\ e \sqsubseteq \mathsf{inl}\ \mathsf{t}$ implies $\mathsf{raise}\ e \leqslant x$ for all $x \in B$
  (c) $\mathsf{inl}\ \mathsf{t} \sqsubseteq \mathsf{inr}\ e$ implies $x \leqslant \mathsf{raise}\ e$ for all $x \in B$
  (d) $\mathsf{inr}\ e \sqsubseteq \mathsf{inr}\ e'$ implies $\mathsf{raise}\ e \leqslant \mathsf{raise}\ e'$.

(2) Let E be a finite set.
- A *preordered semilattice with E-errors* $\underline{B}$ is a preordered set $(X, \leqslant)$ equipped with a monotone binary operation or that is commutative, associative and idempotent, and a function $E \xrightarrow{\mathsf{raise}} X$.
- Such a $\underline{B}$ is *compatible* with a NDBP for E-errors $\sqsubseteq$ when[1]
  (a) $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ implies $x\ \mathsf{orraise}\ D \leqslant x\ \mathsf{orraise}\ D'$ for all $x \in X$
  (b) $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ implies $x\ \mathsf{orraise}\ D \leqslant x\ \mathsf{or}\ y\ \mathsf{orraise}\ D'$ for all $x, y \in X$
  (c) $\{\mathsf{t}, \mathsf{f}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ implies $x\ \mathsf{or}\ y\ \mathsf{orraise}\ D \leqslant x\ \mathsf{orraise}\ D'$ for all $x, y \in X$.
  (d) $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{f}\} + D'$ implies $x\ \mathsf{orraise}\ D \leqslant y\ \mathsf{orraise}\ D'$ for all $x, y \in X$.
- An *I-ary join operation* for $\underline{B}$ (where I is a set) is a monotone function $X^I \xrightarrow{\mathsf{choose}} X$ satisfying neutrality if $I = \emptyset$, and absorption, idempotency and distributivity otherwise.
- If $\underline{B}$ has an ω-ary join operation, it is a *preordered ω-semilattice with E-errors.*

Remark 2.13. In Def. 2.12(2), requirement (2d) is redundant, because, by Lemma 2.7,

$$\begin{array}{rcl} x\ \mathsf{orraise}\ D & \leqslant & x\ \mathsf{or}\ y\ \mathsf{orraise}\ D \\ & \leqslant & y\ \mathsf{orraise}\ D \end{array}$$

To illustrate Def. 2.12(2), consider the three consistent NDBPs for no errors—Fig. 4. A preordered semilattice $(X, \leqslant, \mathsf{or})$ is compatible with

[1] For $D = \{e_0, \ldots, e_{n-1}\}$, we write $x\ \mathsf{orraise}\ D$ as an abbreviation for $x\ \mathsf{or}\ \mathsf{raise}\ e_0\ \mathsf{or}\ \cdots\ \mathsf{or}\ \mathsf{raise}\ e_{n-1}$.

- EQUALITY always
- INCLUSION when it is *inflationary* i.e. when $\forall x, y \in X.\ x \leqslant x \mathsf{\ or\ } y$, or equivalently when or is a least upper bound operator wrt $\leqslant$
- REFINEMENT when it is *deflationary*, the dual property.

Next, consider the four consistent NDBPs for divergence that are pointed (i.e. have divergence as a least element)—Fig. 5. A preordered semilattice with $\{\mathsf{d}\}$-errors $(X, \leqslant, \mathsf{or}, \mathsf{raise})$ is compatible with

- CONVEX when $\perp \stackrel{\text{def}}{=} \mathsf{raise\ d}$ is least
- SMASH when $\perp$ is least and *chaotic* i.e. $\forall x \in X.\ x \mathsf{\ or\ } \perp = \perp$
- UPPER when $\perp$ is least and or is deflationary—this implies that $\perp$ is chaotic
- LOWER when $\perp$ is least and or is inflationary—the second condition is equivalent to $\perp$ being neutral.

Def. 2.12 is justified by the following result. We shall see below (Prop. 4.7) that, for each part, the converse also holds.

Lemma 2.14.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E. Let $\underline{B} = (X, \leqslant, \mathsf{raise})$ be a preordered set with E-errors, such that*

$$(\mathbb{B} + E, \sqsubseteq) \times (X, \leqslant)^{\mathbb{B}} \xrightarrow{\mathsf{Dif}^{E}_{\underline{B}}} (X, \leqslant)$$

is monotone. Then $\underline{B}$ is compatible with $\sqsubseteq$.

(2) *Let E be a finite set and let $\sqsubseteq$ be e a NDBP for E. Let $\underline{B} = (X, \leqslant, \mathsf{or}, \mathsf{raise})$ be a preordered semilattice with E-errors, such that*

$$(\mathcal{P}^{>0}(\mathbb{B} + E), \sqsubseteq) \times (X, \leqslant)^{\mathbb{B}} \xrightarrow{\mathsf{NDif}^{E}_{\underline{B}}} (X, \leqslant)$$

is monotone. Then $\underline{B}$ is compatible with $\sqsubseteq$.

Proof.

(1) Compatibility is proved as follows.
  (a) If $\mathsf{inl\ t} \sqsubseteq \mathsf{inl\ f}$ then apply $\mathsf{Dif}^{E}_{\underline{B}}(-, \{\mathsf{t}.x, \mathsf{f}.y\})$ giving $x \leqslant y$.
  (b) If $\mathsf{inr}\ e \sqsubseteq \mathsf{inl\ t}$ then apply $\mathsf{Dif}^{E}_{\underline{B}}(-, \{\mathsf{t}.x, \mathsf{f}.x\})$ giving $\mathsf{raise}\ e \leqslant x$.
  (c) Dual.
  (d) If $\mathsf{inr}\ e \sqsubseteq \mathsf{inr}\ e'$, then apply $\mathsf{Dif}^{E}_{\underline{B}}(-, \{\mathsf{t}.\mathsf{raise}\ e, \mathsf{f}.\mathsf{raise}\ e\})$ giving $\mathsf{raise}\ e \leqslant \mathsf{raise}\ e'$.

(2) Compatibility is proved as follows.
  - If $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ then apply $\mathsf{NDif}^{E}_{\underline{B}}(-, \{\mathsf{t}.x, \mathsf{f}.x\})$ giving $x \mathsf{\ orraise\ } D \leqslant x \mathsf{\ orraise\ } D'$.
  - If $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ then apply $\mathsf{NDif}^{E}_{\underline{B}}(-, \{\mathsf{t}.x, \mathsf{f}.y\})$ giving $x \mathsf{\ orraise\ } D \leqslant x \mathsf{\ or\ } y \mathsf{\ orraise\ } D'$.
  - Dual. □

If $\leqslant$ is a precongruence of interest on our language $\mathcal{L}$, then at any type τ, the terms modulo $\leqslant$ form a partially ordered semilattice (or partially ordered set) with E-errors. By Lemma 2.14, it must be compatible with the NDBP (or DBP) provided by $\leqslant$.

2.3. **Ordering Terms of Ground Type.** The reader might suspect that our focus on the boolean type is unjustified. Perhaps a ground type with more values, such as an integer type, would give a bigger range of preorders? Fortunately that is not the case. We shall see the boolean ordering determines the ordering at every ground type.

Definition 2.15.

(1) Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. For any set A, we define a relation $\sqsubseteq_A$ on $A + E$ as follows.
 (a) $\mathsf{inl}\ a \sqsubseteq_A \mathsf{inl}\ b$ when $a = b$ or $\mathsf{inl}\ \mathsf{t} \sqsubseteq \mathsf{inl}\ \mathsf{f}$.
 (b) $\mathsf{inr}\ e \sqsubseteq_A \mathsf{inl}\ b$ when $\mathsf{inr}\ e \sqsubseteq \mathsf{inl}\ \mathsf{t}$.
 (c) $\mathsf{inl}\ a \sqsubseteq_A \mathsf{inr}\ e$ when $\mathsf{inl}\ \mathsf{t} \sqsubseteq \mathsf{inr}\ e$.
 (d) $\mathsf{inr}\ e \sqsubseteq_A \mathsf{inr}\ e'$ when $\mathsf{inr}\ e \sqsubseteq \mathsf{inr}\ e'$.

(2) Let E be a finite set, and let $\sqsubseteq$ be NDBP for E-errors. For any set A, we define a relation $\sqsubseteq_A$ on $\mathcal{P}(A + E)$, setting $K + D \sqsubseteq_A K' + D'$ when all the following conditions are met.
 (a) $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$.
 (b) if $\{\mathsf{t}, \mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$ then $K \subseteq K'$.
 (c) if $\{\mathsf{t}\} + D \not\sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ then $K' \subseteq K$.

This construction enjoys the following properties. We defer the proof until after Prop. 4.7.

Proposition 2.16.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. Write $T : Z \mapsto Z + E$.*
 - *For any set A, $\sqsubseteq_A$ is a preorder on TA.*
 - *For any sets A and B, the following is monotone:*

$$(TA, \sqsubseteq_A) \times (TB, \sqsubseteq_B)^A \xrightarrow{\mathsf{Dcase}^E_{A,B}} (TB, \sqsubseteq_B)$$

 - *$\sqsubseteq_{\mathbb{B}}$ is precisely $\sqsubseteq$.*
 - *For any injection $A \xrightarrow{i} B$, the preorder $\sqsubseteq_A$ is $\sqsubseteq_B$ restricted along Ti.*

(2) *Let E be a set, and let $\sqsubseteq$ be a NDBP for E-errors. Write $T : Z \mapsto Z + E$.*
 (a) *For any set A, $\sqsubseteq_A$ is a preorder on TA.*
 (b) *For any sets A and I, the following is monotone:*

$$(TA, \sqsubseteq_A)^I \xrightarrow{\bigcup_I} (TA, \sqsubseteq_A)$$

 (c) *For any sets A and B, the following is monotone:*

$$(TA, \sqsubseteq_A) \times (TB, \sqsubseteq_B)^A \xrightarrow{\mathsf{NDcase}^E_{A,B}} (TB, \sqsubseteq_B)$$

 (d) *$\sqsubseteq_{\mathbb{B}}$ is the unique extension of $\sqsubseteq$ to a preorder on $T\mathbb{B}$ making $\mathsf{NDcase}^E_{\mathbb{B},\mathbb{B}}$ and $\cup$ monotone.*
 (e) *For any injection $A \xrightarrow{i} B$, the preorder $\sqsubseteq_A$ is $\sqsubseteq_B$ restricted along Ti.*

Suppose that σ is a ground type of our language $\mathcal{L}$, and C is a finite set of values. If $\mathcal{L}$ is a deterministic language, then for any term $M : \sigma$ we write

$$\texttt{Dchar}_C\ M \stackrel{\text{def}}{=} \texttt{case}\ M\ \texttt{of}\ \{a.\ \texttt{true}\}_{a \in C}\ \texttt{else false : bool}$$

If $\mathcal{L}$ is nondeterministic, then for any term $M : \sigma$ we write

$$\texttt{NDchar}_C\ M \stackrel{\text{def}}{=} \texttt{false or case}\ M\ \texttt{of}\ \{a.\ \texttt{true}\}_{a \in C}\ \texttt{else false : bool}$$

The semantic equivalents of these constructions are as follows.

Definition 2.17. Let E be a set. Let A be a set and let $C \subseteq A$ be finite. We define functions

$$\begin{array}{rcccl} \mathsf{Dchar}^{E}_{A,C} & : & A+E & \longrightarrow & \mathbb{B}+E \\ & & x & \mapsto & \mathsf{Dcase}^{E}_{A,\mathbb{B}}(x, \{a \in C.\, \mathsf{inr}\ \mathsf{t}, a \notin C.\, \mathsf{inr}\ \mathsf{f}\}) \end{array}$$

$$\begin{array}{rcccl} \mathsf{NDchar}^{E}_{A,C} & : & \mathcal{P}(A+E) & \longrightarrow & \mathcal{P}^{>0}(\mathbb{B}+E) \\ & & x & \mapsto & \{\mathsf{inl}\ \mathsf{f}\} \cup \mathsf{NDcase}^{E}_{A,\mathbb{B}}(x, \{a \in C.\, \{\mathsf{inl}\ \mathsf{t}\}, a \notin C.\, \{\mathsf{inr}\ \mathsf{f}\}\}) \end{array}$$

Lemma 2.18.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. Let A be a set, and let $x, y \in A+E$. If $x \not\sqsubseteq_A y$ then there is a finite $C \subseteq A$ such that*
$$\mathsf{Dchar}^{E}_{A,C}(x) \not\sqsubseteq \mathsf{Dchar}^{E}_{A,C}(y) \tag{2.1}$$

(2) *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. Let A be a set, and let $x, y \in \mathcal{P}(A+E)$. If $x \not\sqsubseteq_A y$ then there is a finite $C \subseteq A$ such that*
$$\mathsf{NDchar}^{E}_{A,C}(x) \not\sqsubseteq \mathsf{NDchar}^{E}_{A,C}(y) \tag{2.2}$$

Proof.

(1) Suppose $x \not\sqsubseteq_A y$, and consider the cases.
   - If $x = \mathsf{inl}\ a$ and $y = \mathsf{inl}\ a'$, we put $C \stackrel{\text{def}}{=} \{a\}$. Since $a \neq a'$, (2.1) reduces to $\mathsf{inl}\ \mathsf{t} \not\sqsubseteq \mathsf{inl}\ \mathsf{f}$, which must be the case for otherwise $x \sqsubseteq_A y$.
   - If $x = \mathsf{inr}\ e$ and $y = \mathsf{inl}\ a$, we put $C \stackrel{\text{def}}{=} \{a\}$. Then (2.1) reduces to $\mathsf{inr}\ e \not\sqsubseteq \mathsf{inl}\ \mathsf{t}$, which must be the case for otherwise $x \sqsubseteq_A y$.
   - The case $x = \mathsf{inl}\ a$ and $y = \mathsf{inr}\ e$ is treated dually.
   - If $x = \mathsf{inr}\ e$ and $y = \mathsf{inr}\ e'$, we put $C \stackrel{\text{def}}{=} \{\}$. Then (2.1) reduces to $\mathsf{inr}\ e \not\sqsubseteq \mathsf{inr}\ e'$, which must be the case for otherwise $x \sqsubseteq_A y$.

(2) Put $x = K + D$ and $y = K' + D'$. Supposing $x \not\sqsubseteq_A y$, one of the following must hold.
   - $\{\mathsf{t}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$. In this case we put $C \stackrel{\text{def}}{=} \{\}$, so (2.2) reduces to $\{\mathsf{f}\} + D \not\sqsubseteq \{\mathsf{f}\} + D'$ which is correct.
   - $\{\mathsf{t}, \mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$ and $K \not\subseteq K'$. In this case we pick $a \in K \setminus K'$ and put $C \stackrel{\text{def}}{=} \{a\}$, so (2.2) reduces to $\{\mathsf{t}, \mathsf{f}\} + D \not\sqsubseteq \{\mathsf{f}\} + D'$ which is correct.
   - $\{\mathsf{t}\} + D \not\sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ and $K' \not\subseteq K$. Dual argument. □

Proposition 2.19.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. Let A be a set. Then any preorder $\leqslant$ on $A+E$ making the functions*

$$(\mathbb{B}+E, \sqsubseteq) \times (A+E, \leqslant)^{\mathbb{B}} \xrightarrow{\mathsf{Dif}^{E}_{A}} (A+E, \leqslant)$$
$$(A+E, \leqslant) \xrightarrow{\mathsf{Dchar}^{E}_{A,C}} (\mathbb{B}+E, \sqsubseteq) \quad \textit{for every finite } C \subseteq A$$

monotone must be $\sqsubseteq_A$.

(2) *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. Let A be a set, and let $\mathcal{U}$ be a subset of $\mathcal{P}(A+E)$ that contains all singleton sets and is closed under binary union. Then any preorder $\leqslant$ on $\mathcal{U}$ making the functions*

$$(\mathcal{P}^{>0}(\mathbb{B}+E),\sqsubseteq)\times(\mathcal{U},\leqslant)^{\mathbb{B}} \xrightarrow{\mathsf{NDif}^{E}_{A}} (\mathcal{U},\leqslant)$$

$$(\mathcal{U},\leqslant) \xrightarrow{\mathsf{NDchar}^{E}_{A,C}} (\mathcal{P}^{>0}(\mathbb{B}+E),\sqsubseteq) \quad \textit{for every finite } C\subseteq A$$

monotone must be $\sqsubseteq_A$ restricted to $\mathcal{U}$.

Proof.

(1) To prove $\sqsubseteq_A$ is contained in $\leqslant$, we first see that $(A+E,\leqslant,\mathsf{inr})$ is compatible with $\sqsubseteq$, by Lemma 2.14(1). Then we consider the cases.
- If $\mathsf{inl}\ a \sqsubseteq_A \mathsf{inl}\ a'$ then either $a=a'$ or $\sqsubseteq$ is inconsistent. Either way, $\mathsf{inl}\ a\leqslant \mathsf{inl}\ a'$, by compatibility.
- If $\mathsf{inr}\ e\sqsubseteq_A \mathsf{inl}\ a$ then $\mathsf{inr}\ e\sqsubseteq \mathsf{inl}\ \mathsf{t}$ so $\mathsf{inr}\ e\leqslant \mathsf{inl}\ a$ by compatibility.
- Dually if $\mathsf{inl}\ a\sqsubseteq_A \mathsf{inr}\ e$.
- If $\mathsf{inr}\ e\sqsubseteq_A \mathsf{inr}\ e'$ then $\mathsf{inr}\ e\sqsubseteq \mathsf{inr}\ e'$ so $\mathsf{inr}\ e\leqslant \mathsf{inr}\ e'$ by compatibility.

Lemma 2.18(1) tells us that $\leqslant$ is contained in $\sqsubseteq_A$.

(2) To prove $\sqsubseteq_A\restriction\mathcal{U}$ is contained in $\leqslant$, we first see that $(\mathcal{U},\leqslant,\mathsf{inr})$ is compatible with $\sqsubseteq$, by Lemma 2.14(2). Suppose $K+D\sqsubseteq_A K'+D'$, so $\{\mathsf{t}\}+D\sqsubseteq\{\mathsf{t}\}+D'$. We show

$$\begin{aligned} K+D &\leqslant (K\cup K')+(D\cup D') && (2.3)\\ &\leqslant K'+D' && (2.4)\end{aligned}$$

We prove only (2.3), as (2.4) is proved dually.
- If $\{\mathsf{t}\}+D\sqsubseteq\{\mathsf{t},\mathsf{f}\}+D'$ then compatibility gives

$$\begin{aligned} K+D &= (K+D)\cup(\emptyset+D)\\ &\leqslant (K+D)\cup(K'+D')\cup(\emptyset+D')\\ &= (K\cup K')+(D\cup D')\end{aligned}$$

- Otherwise $K'\subseteq K$ and $\{\mathsf{t}\}+D\sqsubseteq\{\mathsf{t}\}+D'$ so compatibility gives

$$\begin{aligned} K+D &= (K+D)\cup(\emptyset+D)\\ &\sqsubseteq (K+D)\cup(\emptyset+D')\\ &= (K\cup K')+(D\cup D')\end{aligned}$$

Lemma 2.18(2) tells us that $\leqslant$ is contained in $\sqsubseteq_A$. □

If $\leqslant$ is a precongruence of interest on our language $\mathcal{L}$, then Prop. 2.19 tells us that the ordering at each ground type is determined by the NDBP (or DBP) provided by $\leqslant$.

Remark 2.20. A similar analysis is given in [AP97], Thm. 17 (attributed to Gordon Plotkin), in the setting of NDBCs for divergence. Our generalization is to consider preorders rather than equivalence relations.

INCONSISTENT
EQUALITY
INCLUSION
REFINEMENT
LOWER
UPPER
SESQUI INCLUSION
SESQUI REFINEMENT
LOWER CONGRUENCE

Figure 7: All the ABPs for no errors

2.4. **Ambiguous Nondeterminism.** We recall McCarthy's amb operator: M `amb` M' evaluates both M and M' on an arbitrary fair scheduler, then returns whatever it gets first. It diverges only if both M and M' diverge. The semantic I-ary version of this is as follows.

Definition 2.21.

(1) Let A and I be sets. We write amb^I_A for the I-ary operation

$$\begin{array}{ccl} (\mathcal{P}\, A_\bot)^I & \longrightarrow & \mathcal{P}\, A_\bot \\ x & \mapsto & \{\mathsf{up}\ a \mid \exists i \in I.\, \mathsf{up}\ a \in x_i\} \cup \{\bot \mid \forall i \in I.\, \bot \in x_i\} \end{array}$$

where $A_\bot \stackrel{\text{def}}{=} A + \{\mathsf{d}\}$. We write amb_A (infix) for the binary case.

(2) Let E be a finite set. An *amb boolean precongruence* (ABP) for E-errors is a preorder on $\mathcal{P}(\mathbb{B}+E)_\bot$ that is a NDBP for divergence and E-errors and such that

$$\begin{array}{ccc} (\mathcal{P}^{>0}\,(\mathbb{B}+E)_\bot, \sqsubseteq)^2 & \longrightarrow & (\mathcal{P}^{>0}\,(\mathbb{B}+E)_\bot, \sqsubseteq) \\ x, y & \mapsto & x\ \mathsf{amb}_{\mathbb{B}}\ y \end{array}$$

is monotone. If symmetric, it is an *amb boolean congruence* (ABC) for E-errors.

Remark 2.22. If E is a countable set, then we can still use Def. 2.21(2) provided we additionally require countable amb to be monotone.

The ABPs for no errors are listed in Fig. 7. All of them are intersections of LOWER and INCLUSION and their converses. We note that the only consistent ABP making divergence least is LOWER; this observation was first made in [LLP05].

Once again, the boolean type is sufficient for all ground types.

Proposition 2.23. *Let E be a finite set, and let $\sqsubseteq$ be an ABP for E-errors. Then the function*

$$(\mathcal{P}\, A_\bot, \sqsubseteq_A)^I \xrightarrow{\mathsf{amb}^I_A} (\mathcal{P}\, A_\bot, \sqsubseteq_A)$$

is monotone for any sets A and I.

We omit the proof, as we give a more general theorem below—Lemma 4.4(2).

Remark 2.24. Deterministic parallel operators such as parallel-or and parallel-exists [Plo77] do not cause the same problems as amb. They are monotone wrt all of the DBPs and NDBPs for divergence (assuming the empty set is excluded).

3. Contexts and Tests

3.1. **Contextual Preorders.** It is easy to see that any NDBP (or DBP) $\sqsubseteq$ induces a *contextual preorder* on our language $\mathcal{L}$. Two terms $M, M' : \tau$ are *contextually related* up to $\sqsubseteq$, when for every context

$$\cdot : \tau \vdash \mathcal{C}[\cdot] : \sigma$$

where σ is a ground type whose set of values is $A(\sigma)$, the meanings of $\mathcal{C}[M]$ and $\mathcal{C}[M']$ are related by $\sqsubseteq_{A(\sigma)}$. Equivalently—by Lemma 2.18—we can consider only contexts of boolean type. Prop. 2.19 tells us that any preorder determined by contexts of ground type must arise in this way.

We note some preorders appearing in the denotational literature that turn out to be contextual preorders.

In [Lai07a], a deterministic language with divergence and crash is studied, and a model described using three orderings $\leq^E, \leq^B, \updownarrow$. These are in fact the contextual preorders induced, respectively, by BIPOINTED, BISTABLE and BISTABLE COHERENCE (Fig. 3). Similarly in [Lai06, Lai07b] a nondeterministic language with divergence and crash is studied, and two models are studied, each using two orders. These are the contextual preorders induced by MAY, STABLE, MUST and COSTABLE (Fig. 6).

CSP is a well-known process calculus with two erroneous behaviours: divergence and deadlock. Various models of CSP are described in [Ros98, Ros93]:

- $\mathcal{T}$, where a process denotes its finite trace set
- $\mathcal{F}$, where a process denotes its finite trace set and stable failure set
- $\mathcal{I}$, where a process denotes its finite trace set, divergence set and infinite trace set, each saturated with extensions of divergences
- $\mathcal{U}$, where a process denotes its finite trace set (redundant, therefore usually omitted), stable failure set, divergence set and infinite trace set, each saturated with extensions of divergences.

For $\mathcal{I}$ and $\mathcal{U}$, as well as refinement, a stronger ordering called "definedness" is studied [Ros92]. That gives six ordered models, and they can be regarded as contextual preorders for the NDBPs shown in Fig. 8. Although CSP does not have a boolean type in our sense, the full abstraction results presented in [Ros98] for the equivalences can be adapted to the preorders.

Curiously, the three rows correspond to LOWER, UPPER and SMASH, which are NDBPs for divergence alone. Each column extends these to deadlock in a uniform way: the left column by making deadlock a neutral element, the right column by treating it as a value.

3.2. **Intersection Theorems.** An intersection theorem says that every precongruence or congruence is an intersection of certain special ones. We have already seen examples of this in Sect. 2.1 and Sect. 2.4. In the deterministic case, there is a general such theorem, using the following special precongruences.

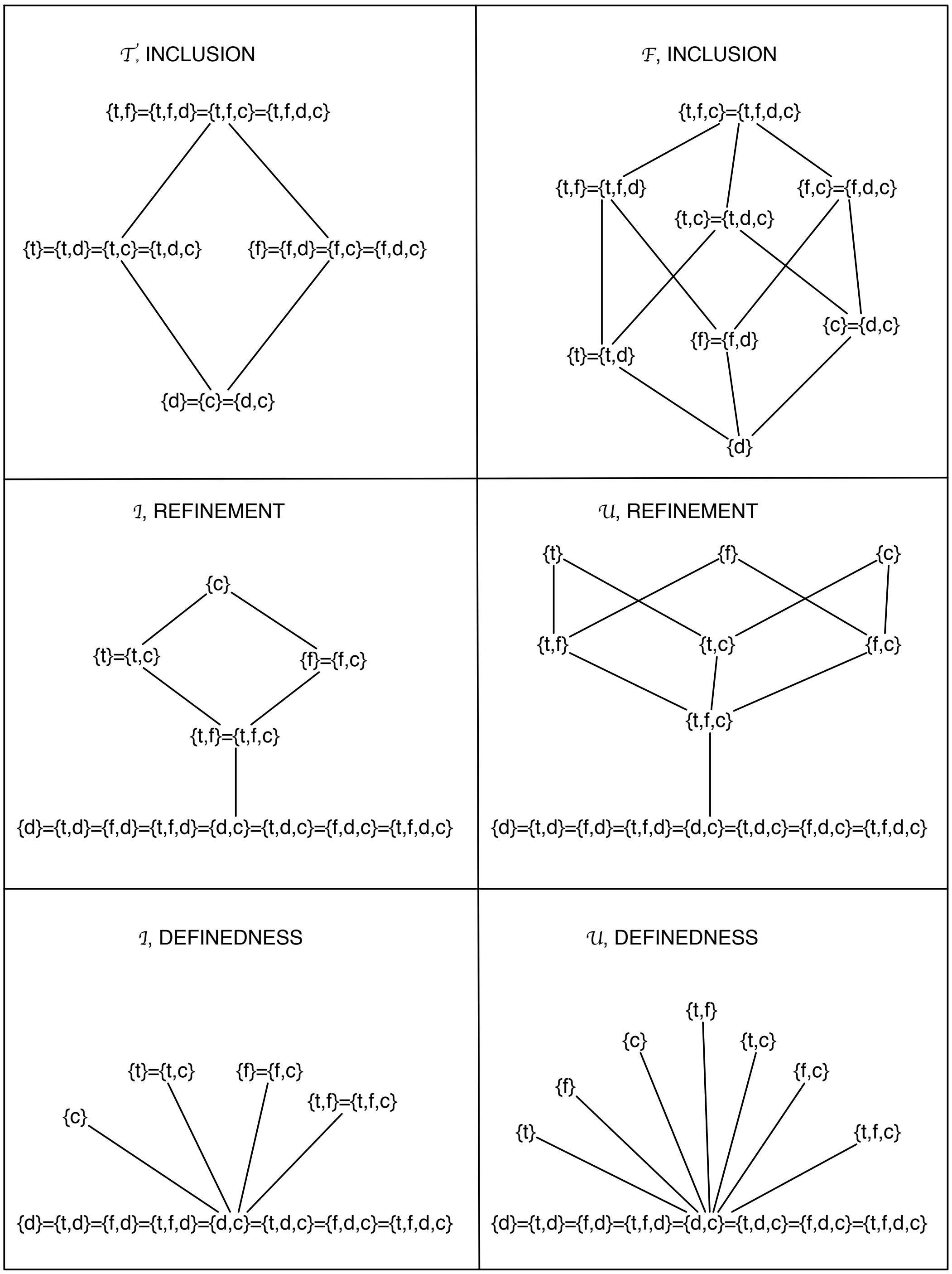

Figure 8: NDBPs for divergence (d) and deadlock (c) in standard CSP models

Definition 3.1. Let E be a set, and let $C \subseteq E$. We write $\lesssim^{(C)}$ for the DBP for E-errors given by

$$\begin{tikzcd}
 & e' \arrow[dl, dash] \arrow[dr, dash] & \\
\mathsf{t} & & \mathsf{f} \\
 & e \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

where $e \in C$ and $e' \in E \setminus C$. We write $\equiv^{(C)}$ for its symmetrization.

A consistent DBP is of the form $\lesssim^{(C)}$, for some $C \subseteq E$, iff every error is either least or greatest. These precongruences give the following intersection theorem.

Proposition 3.2. *Let E be a set.*

(1) *Any DBP for E-errors is of the form $\bigcap_{C \in \mathcal{C}} \lesssim^{(C)}$, for some $\mathcal{C} \subseteq \mathcal{P}E$.*
(2) *Any DBC for E-errors is of the form $\bigcap_{C \in \mathcal{C}} \equiv^{(C)}$, for some $\mathcal{C} \subseteq \mathcal{P}E$.*

Proof. (1) Let $\sqsubseteq$ be a DBP for E-errors. Given $x \not\sqsubseteq y$, we need to find $C \subseteq E$ such that $\lesssim^{(C)}$ contains $\sqsubseteq$ and $x \not\lesssim^{(C)} y$. We set $C \stackrel{\text{def}}{=} \{e \in E \mid \mathsf{inr}\ e \sqsubseteq y\}$.

If $x \lesssim^{(C)} y$, then either
- $x = y$, implying $x \sqsubseteq y$, a contradiction
- $x = \mathsf{inr}\ e$ with $e \in C$, giving $x \sqsubseteq y$, a contradiction
- $y = \mathsf{inr}\ e$ with $e \in E \setminus C$, giving $y \not\sqsubseteq y$, a contradiction.

So $x \not\lesssim^{(C)} y$. We show $\lesssim^{(C)}$ contains $\sqsubseteq$ as follows.
- Suppose $\mathsf{inl}\ b \sqsubseteq \mathsf{inl}\ b'$. Then $b = b'$ since $\sqsubseteq$ is consistent, so $\mathsf{inl}\ b \lesssim^{(C)} \mathsf{inl}\ b'$.
- Suppose $\mathsf{inr}\ e \sqsubseteq \mathsf{inl}\ b$. Applying $\mathsf{Dif}^{E}_{\mathbb{B}}(-, \{i.y\})$ gives $\mathsf{inr}\ e \sqsubseteq y$. So $e \in C$, so $\mathsf{inr}\ e \lesssim^{(C)} \mathsf{inl}\ b$.
- Suppose $\mathsf{inl}\ b \sqsubseteq \mathsf{inr}\ e$. Applying $\mathsf{Dif}^{E}_{\mathbb{B}}(-, \{i.x\})$ gives $x \sqsubseteq \mathsf{inr}\ e$. If $e \in C$ then $x \sqsubseteq \mathsf{inr}\ e \sqsubseteq y$, a contradiction. So $e \in E \setminus C$, so $\mathsf{inl}\ b \lesssim^{(C)} \mathsf{inr}\ e$.
- Suppose $\mathsf{inr}\ e \sqsubseteq \mathsf{inr}\ e'$. If $e' \in C$ then $\mathsf{inr}\ e \sqsubseteq \mathsf{inr}\ e' \sqsubseteq y$ so $e \in C$. So $\mathsf{inr}\ e \lesssim^{(C)} \mathsf{inr}\ e'$.

(2) Obvious.

We do not give a general result for NDBPs, although it might be possible to do so. Instead we restrict our attention to inflationary, deflationary and symmetric ones. These special classes are related as follows.

Proposition 3.3. *Let E be a finite set. Consider the sets and functions*

$$\begin{tikzcd}
\begin{array}{c}\textit{inflationary}\\ \textit{NDBPs}\\ \textit{with } E\textit{-errors}\end{array} \arrow[rr, leftrightarrow, "\textit{converse}"] \arrow[dr, "\textit{symmetrization}"'] & & \begin{array}{c}\textit{deflationary}\\ \textit{NDBPs}\\ \textit{with } E\textit{-errors}\end{array} \arrow[dl, "\textit{symmetrization}"] \\
 & \begin{array}{c}\textit{NDBCs}\\ \textit{with } E\textit{-errors}\end{array} &
\end{tikzcd}$$

(1) *The intersection of any two is* {INCONSISTENT}.
(2) *The functions are bijections and preserve intersections.*

The same result holds for ABPs for E-errors.

Proof. The only nontrivial part is showing that any NDBC for E-errors $\equiv$ is the symmetrization of a unique inflationary NDBP for E-errors $\sqsubseteq$.

Let $\sqsubseteq$ be $\{(x, y) \mid x \cup y \equiv y\}$, clearly the only possibility.

Reflexivity is trival. If $x \sqsubseteq y$ and $y \sqsubseteq z$ then

$$\begin{array}{rcl} z & \equiv & y \cup z \\ & \equiv & x \cup y \cup z \\ & \equiv & x \cup z \end{array}$$

Clearly $\mathsf{NDif}^E_{\mathbb{B}}$ is monotone wrt $\sqsubseteq$ in its first argument. Let $g, g' \in (\mathcal{P}^{>0}(bools + E))^{\mathbb{B}}$ be such that $gb \sqsubseteq g'b$ for all $b \in \mathbb{B}$. Then for $u \in \mathcal{P}^{>0}(\mathbb{B} + E)$ we have

$$\begin{array}{rcl} \mathsf{NDif}^E_{\mathbb{B}}(u, \{b.g'b\}) & \equiv & \mathsf{NDif}^E_{\mathbb{B}}(u, \{b.gb \text{ or } g'b\}) \\ & = & \mathsf{NDif}^E_{\mathbb{B}}(u, \{b.gb\}) \text{ or } \mathsf{NDif}^E_{\mathbb{B}}(u, \{b.g'b\}) \end{array}$$

So $\sqsubseteq$ is a NDBP for E-errors. Clearly it is inflationary and its symmetrization is $\equiv$.

If $\equiv$ is an ABC, we also must show that amb is monotone wrt $\sqsubseteq$. Monotonicity in one argument suffices. If $x \sqsubseteq y$ then

$$\begin{array}{rcl} y \;\mathsf{amb}_{\mathbb{B}}\; u & \equiv & (x \cup y) \;\mathsf{amb}_{\mathbb{B}}\; u \\ & = & (x \;\mathsf{amb}_{\mathbb{B}}\; u) \cup (y \;\mathsf{amb}_{\mathbb{B}}\; u) \end{array}$$

□

In an inflationary NDBP, an error is least iff neutral, and greatest iff chaotic. We accordingly consider the following special precongruences.

Definition 3.4. Let E be a finite set, and let $C \subseteq E$. We write $\lesssim^{\mathsf{ND}(C)}$ for the inflationary DBP for E-errors given by

$$\begin{tikzcd}[column sep=small] & \emptyset + D' = \{\mathsf{t}\} + D' = \{\mathsf{f}\} + D' = \{\mathsf{t}, \mathsf{f}\} + D' \arrow[d, no head] & \\ & \{\mathsf{t}, \mathsf{f}\} + D & \\ \{\mathsf{t}\} + D \arrow[ur, no head] & & \{\mathsf{f}\} + D \arrow[ul, no head] \\ & \emptyset + D \arrow[ul, no head] \arrow[ur, no head] & \end{tikzcd}$$

where $D \subseteq C$ and $D' \not\subseteq C$. We write $\equiv^{\mathsf{ND}(C)}$ for its symmetrization.

An inflationary consistent NDBP is of the form $\lesssim^{\mathsf{ND}(C)}$ iff every error is either least (neutral) or greatest (chaotic). Similarly, a consistent NDBC is of the form $\equiv^{\mathsf{ND}(C)}$ iff every error is either neutral or chaotic. We can now formulate our intersection theorem.

Proposition 3.5. *Let E be a finite set.*

(1) *Any inflationary NDBP for E-errors is of the form $\bigcap_{C \in \mathcal{C}} \lesssim^{\mathsf{ND}(C)}$ for some $C \subseteq \mathcal{P}E$.*

(2) *Any NDBC for E-errors is of the form $\bigcap_{C\in\mathcal{C}} \equiv^{\mathsf{ND}(C)}$ for some $\mathcal{C} \subseteq E$.*

Proof. (1) Let $\sqsubseteq$ be an inflationary NDBP for E-errors. It suffices to show that, given $K+D \not\sqsubseteq K'+D'$, there is some $C \subseteq E$ such that $\lesssim^{\mathsf{ND}(C)}$ contains $\sqsubseteq$ and $K + D \not\lesssim^{\mathsf{ND}(C)} K'+D'$. We set $C \stackrel{\text{def}}{=} \{e \in E \mid K' + (D' \cup \{e\}) \sqsubseteq K' + D'\}$. Thus $L + F \lesssim^{\mathsf{ND}(C)} L' + F'$ iff either

(a) there is $e \in F'$ such that $K' + (D' \cup \{e\}) \not\sqsubseteq K' + D'$, or
(b) $K' + (D' \cup F) \sqsubseteq K' + D'$ and $L \subseteq L'$.

If $K+D \lesssim^{\mathsf{ND}(C)} K'+D'$ then, since (1a) cannot be true, (1b) must be, i.e. $K \subseteq K'$ and $K' + (D \cup D') \sqsubseteq K' + D'$. Hence

$$\begin{array}{rcl} K + D & \subseteq & (K \cup K') + (D \cup D') \\ & = & K' + (D \cup D') \\ & \sqsubseteq & K' + D' \end{array}$$

a contradiction. So $K + D \not\lesssim^{\mathsf{ND}(C)} K' + D'$.

We show $\lesssim^{\mathsf{ND}(C)}$ contains $\sqsubseteq$ as follows. Suppose $L+F \sqsubseteq L'+F'$ but $L+F \not\lesssim^{\mathsf{ND}(C)} L' + F'$.

The failure of (1a) tells us, for all $e \in F'$, that $K' + (D' \cup \{e\}) \sqsubseteq K' + D'$. Taking the union of all these, along with $K' + D' \sqsubseteq K' + D'$ (to ensure the union is nonempty), gives $K' + (D' \cup F') \sqsubseteq K' + D'$.

Applying $(K'+D') \cup \mathsf{NDif}^E_{\mathbb{B}}(-, i.K' + D')$ to $L+F \sqsubseteq L'+F'$ gives $K'+(D' \cup F) \sqsubseteq K' + (D' \cup F')$ and hence $K' + (D' \cup F) \sqsubseteq K' + D'$.

Hence the failure of (1b) must be caused by some $b \in L \setminus L'$. Applying $(K'+D') \cup \mathsf{NDif}^E_{\mathbb{B}}(,b.K + D, i \neq b.K' + D')$ to $L+F \sqsubseteq L'+F'$ gives $(K \cup K') + (D \cup D'\ cupF) \sqsubseteq K' + (D' \cup F')$. Hence

$$\begin{array}{rcl} K + D & \subseteq & (K \cup K') + (D \cup D' \cup F) \\ & \sqsubseteq & K' + (D' \cup F') \\ & \sqsubseteq & K' + D' \end{array}$$

a contradiction.

(2) Immediate from (1) and Prop. 3.3. □

3.3. Testing at Sierpinski Type or Zero Type.

We have seen that, to define a contextual preorder, it suffices to consider contexts of boolean type. In many situations—indeed, for all the boolean precongruences we have noted in the literature—one can use contexts of Sierpinski type. This underlies the idea of *testing* preorders [NH84]: a process either passes the test (returns the sole value of Sierpinski type) or does not. And occasionally one can go even further by using only contexts of zero type [Lai07a, Lai06].

In general, these contexts give a weaker preorder. We examine when they do in fact suffice.

Definition 3.6.

(1) Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. For $x, y \in \mathbb{B} + E$ we write

$$\begin{array}{lll} x\ \mathrm{Test}_1(\sqsubseteq)\ y & \text{when} & \mathsf{Dif}^E_1(x,h) \sqsubseteq_1 \mathsf{Dif}^E_1(y,h) \text{ for all } h \in (1+E)^{\mathbb{B}} \\ x\ \mathrm{Test}_0(\sqsubseteq)\ y & \text{when} & \mathsf{Dif}^E_\emptyset(x,h) \sqsubseteq_\emptyset \mathsf{Dif}^E_1(y,h) \text{ for all } h \in (\emptyset+E)^{\mathbb{B}} \end{array}$$

(2) Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-erorrs. For $x, y \in \mathcal{P}^{>0}(\mathbb{B}+E)$ we write

$$\begin{array}{lll} x \ \mathrm{Test}_1(\sqsubseteq)\ y & \text{when} & \mathsf{NDif}^E_1(x,h) \sqsubseteq_1 \mathsf{NDif}^E_1(y,h) \text{ for all } h \in (\mathcal{P}^{>0}(1+E))^{\mathbb{B}} \\ x \ \mathrm{Test}_0(\sqsubseteq)\ y & \text{when} & \mathsf{NDif}^E_\emptyset(x,h) \sqsubseteq_\emptyset \mathsf{NDif}^E_1(y,h) \text{ for all } h \in (\mathcal{P}^{>0}(\emptyset+E))^{\mathbb{B}} \end{array}$$

Proposition 3.7. *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors.*

1. $\mathrm{Test}_1(\sqsubseteq)$ *and* $\mathrm{Test}_0(\sqsubseteq)$ *are NDBPs for E-errors and contain $\sqsubseteq$.*
2. $\mathrm{Test}_1(\sqsubseteq)$ *is contained in* $\mathrm{Test}_0(\sqsubseteq)$*.*
3. $(\mathrm{Test}_1(\sqsubseteq))_1$ *and $\sqsubseteq_1$ coincide, as preorders on $\mathcal{P}^{>0}(1+E)$.*
4. $(\mathrm{Test}_0(\sqsubseteq))_\emptyset$ *and* $(\mathrm{Test}_1(\sqsubseteq))_\emptyset$ *and $\sqsubseteq_\emptyset$ coincide, as preorders on $\mathcal{P}^{>0}(\emptyset+E)$.*

The analogous results hold for any set E and DBP for E-errors $\sqsubseteq$.

Proof. We write TZ for $\mathcal{P}^{>0}(Z+E)$.

(1) Clearly $\mathrm{Test}_1(\sqsubseteq)$ and $\mathrm{Test}_0(\sqsubseteq)$ are preorders containing $\sqsubseteq$. We show $\mathrm{Test}_1(\sqsubseteq)$ is a NDBP; the proof for $\mathrm{Test}_0(\sqsubseteq)$ is similar. Let $x, x' \in T\mathbb{B}$ be such that $x \ \mathrm{Test}_1(\sqsubseteq)\ x'$. Let $g, g' \in (T\mathbb{B})^{\mathbb{B}}$ be such that $gb \ \mathrm{Test}_1(\sqsubseteq)\ g'b$ for $b \in \mathbb{B}$. If $h \in \mathcal{P}^{>0}(1+E)^{\mathbb{B}}$ then $\mathsf{NDif}^E_1(gb,h) \sqsubseteq_1 \mathsf{NDif}^E_1(g'b,h)$ for $b \in \mathbb{B}$ and so

$$\begin{array}{rcl} \mathsf{NDif}^E_1(\mathsf{NDif}^E_{\mathbb{B}}(x,g),h) & = & \mathsf{NDif}^E_1(x,\{b.\mathsf{NDif}^E_1(gb,h)\}) \\ & \sqsubseteq_1 & \mathsf{NDif}^E_1(x',\{b.\mathsf{NDif}^E_1(gb,h)\}) \\ & \sqsubseteq_1 & \mathsf{NDif}^E_1(x',\{b.\mathsf{NDif}^E_1(g'b,h)\}) \\ & = & \mathsf{NDif}^E_1(\mathsf{NDif}^E_{\mathbb{B}}(x',g'),h) \end{array}$$

Thus $\mathsf{NDif}^E_{\mathbb{B}}(x,g) \ \mathrm{Test}_1(\sqsubseteq)\ \mathsf{NDif}^E_{\mathbb{B}}(x',g')$.

(2) Suppose $x \ \mathrm{Test}_1(\sqsubseteq)\ y$ and $h \in (T\emptyset)^{\mathbb{B}}$. Define $g \in (T1)^{\mathbb{B}}$ to be $\{b.\mathsf{NDcase}^E_{\emptyset,1}(hb,\{\})\}$. Then $\mathsf{NDif}^E_1(x,g) \sqsubseteq_1 \mathsf{NDif}^E_1(x',g)$. Pick $p \in T\emptyset$, e.g. $\mathsf{NDif}^E_\emptyset(x,h)$. Then

$$\begin{array}{rcl} \mathsf{NDif}^E_\emptyset(x,h) & = & \mathsf{NDcase}^E_{1,\emptyset}(\mathsf{NDif}^E_1(x,g),\{*.p\}) \\ & \sqsubseteq & \mathsf{NDcase}^E_{1,\emptyset}(\mathsf{NDif}^E_1(x',g),\{*.p\}) \\ & = & \mathsf{NDif}^E_\emptyset(x',h) \end{array}$$

(3) Pick an injection $1 \xrightarrow{i} \mathbb{B}$. Let $x, x' \in T1$ be such that $x(\mathrm{Test}_1(\sqsubseteq))_1 x'$ then $(Ti)x \ \mathrm{Test}_1(\sqsubseteq)\ (Ti)x'$ by Prop. 2.16(2d)–(2e). Pick $p \in T1$, e.g. x. Then

$$\begin{array}{rcl} x & = & \mathsf{NDif}^E_1((Ti)x,\{ia.\{\mathsf{inl}\ a\}, b \notin \mathsf{range}\ i.p\}) \\ & \sqsubseteq_1 & \mathsf{NDif}^E_1((Ti)(x',\{ia.\{\mathsf{inl}\ a\}, b \notin \mathsf{range}\ i.p\}) \\ & = & x' \end{array}$$

(4) Similar.

(5) Trivial. $\square$

DBP $\sqsubseteq$	$\mathrm{Test}_1(\sqsubseteq)$	$\mathrm{Test}_0(\sqsubseteq)$
All the DBPs for no errors (Fig. 1)		
INCONSISTENT	INCONSISTENT	INCONSISTENT
EQUALITY	INCONSISTENT	INCONSISTENT
All the DBPs for divergence (Fig. 2)		
INCONSISTENT	INCONSISTENT	INCONSISTENT
EQUALITY	EQUALITY	INCONSISTENT
POINTED	POINTED	INCONSISTENT
OP-POINTED	OP-POINTED	INCONSISTENT
All the DBPs for divergence and crash (Fig. 3)		
INCONSISTENT	INCONSISTENT	INCONSISTENT
EQUALITY	EQUALITY	
BISTABLE COHERENCE	BISTABLE COHERENCE	INCONSISTENT
STABLE	STABLE	BIPOINTED
OP-STABLE	OP-STABLE	OP-BIPOINTED
COSTABLE	COSTABLE	BIPOINTED
OP-COSTABLE	OP-COSTABLE	OP-BIPOINTED
BISTABLE	BISTABLE	BIPOINTED
OP-BISTABLE	OP-BISTABLE	OP-BIPOINTED
BIPOINTED	BIPOINTED	BIPOINTED
OP-BIPOINTED	OP-BIPOINTED	OP-BIPOINTED
DOUBLE POINTED	DOUBLE POINTED	INCONSISTENT
DOUBLE OP-POINTED	DOUBLE OP-POINTED	INCONSISTENT
Canonical DBPs for E-errors (Def. 3.1) with $\|E\| \geqslant 2$		
INCONSISTENT	INCONSISTENT	INCONSISTENT
EQUALITY	EQUALITY	EQUALITY
$\lesssim^{(\emptyset)}$	$\lesssim^{(\emptyset)}$	INCONSISTENT
$\lesssim^{(E)}$	$\lesssim^{(E)}$	INCONSISTENT
$\lesssim^{(C)}$ $(C \neq \emptyset, E)$	$\lesssim^{(C)}$	$\lesssim^{(C)}$
$\equiv^{(\emptyset)}$	$\equiv^{(\emptyset)}$	INCONSISTENT
$\equiv^{(E)}$	$\equiv^{(E)}$	INCONSISTENT
$\equiv^{(C)}$ $(C \neq \emptyset, E)$	$\equiv^{(C)}$	$\equiv^{(C)}$

Figure 9: 1-testing and 0-testing all the DBPs in this paper

It follows from Prop. 3.7 that Test_1 and Test_0 are closure operators on the poset of NDBPs (or DBPs) for E-errors, and they clearly preserve intersection and converse. Their behaviour on all the boolean precongruences mentioned in this paper is given in Fig. 9–10.

A *1-testable* boolean precongruence is a fixpoint of Test_1, and a *0-testable* one is a fixpoint of Test_0. These correspond to contextual preorders that are definable by contexts of Sierpinski type and of zero type, respectively. As stated above, every boolean precongruence we have found in the literature is 1-testable. Both properties are preserved by converse and intersection, and 0-testability implies 1-testability.

We note that

- all the 1-testable NDBPs for divergence are intersections of LOWER and UPPER and their converses.
- all the 0-testable DBPs for divergence and crash are intersections of BIPOINTED and its converse
- all the 0-testable NDBPs for divergence and crash are intersections of MAY and MUST and their converses.

We provide a variety of conditions for 1- and 0-testability.

Proposition 3.8.

(1) *Let E be a nonempty set. Every DBP for E-errors is 1-testable.*
(2) *Let E be a set, and $\equiv$ a consistent DBC for E-errors. Then $\equiv$ is 0-testable iff there are $e, e' \in E$ such that $e \not\equiv e'$.*

NDBP $\sqsubseteq$	$\mathrm{Test}_1(\sqsubseteq)$	$\mathrm{Test}_0(\sqsubseteq)$
All the NDBPs for no errors (Fig. 4)		
INCONSISTENT	INCONSISTENT	INCONSISTENT
EQUALITY	INCONSISTENT	INCONSISTENT
INCLUSION	INCONSISTENT	INCONSISTENT
REFINEMENT	INCONSISTENT	INCONSISTENT
All the NDBPs for divergence (Fig. 5)		
INCONSISTENT	INCONSISTENT	INCONSISTENT
EQUALITY	EQUALITY	INCONSISTENT
LOWER	LOWER	INCONSISTENT
OP-LOWER	OP-LOWER	INCONSISTENT
UPPER	UPPER	INCONSISTENT
OP-UPPER	OP-UPPER	INCONSISTENT
SMASH	UPPER	INCONSISTENT
OP-SMASH	OP-UPPER	INCONSISTENT
CONVEX	CONVEX	INCONSISTENT
OP-CONVEX	OP-CONVEX	INCONSISTENT
INCLUSION	INCLUSION	INCONSISTENT
REFINEMENT	REFINEMENT	INCONSISTENT
SESQUI INCLUSION	SESQUI INCLUSION	INCONSISTENT
SESQUI REFINEMENT	SESQUI REFINEMENT	INCONSISTENT
PLUCKED	PLUCKED	INCONSISTENT
OP-PLUCKED	OP-PLUCKED	INCONSISTENT
STUNTED	INCLUSION	INCONSISTENT
OP-STUNTED	REFINEMENT	INCONSISTENT
LOWER CONGRUENCE	LOWER CONGRUENCE	INCONSISTENT
UPPER CONGRUENCE	UPPER CONGRUENCE	INCONSISTENT
Assorted NDBPs for divergence and crash/deadlock (Fig. 6, Fig. 8, Fig. 12)		
MAY	MAY	MAY
COSTABLE	COSTABLE	MAY
MUST = $\mathcal{I}$, REFINEMENT	MUST	MUST
STABLE	STABLE	MUST
$\mathcal{T}$, INCLUSION	$\mathcal{T}$, INCLUSION	INCONSISTENT
$\mathcal{F}$, INCLUSION	$\mathcal{F}$, INCLUSION	MAY
$\mathcal{U}$, REFINEMENT	$\mathcal{U}$, REFINEMENT	MUST
$\mathcal{I}$, DEFINEDNESS	$\mathcal{I}$, DEFINEDNESS	MUST
$\mathcal{U}$, DEFINEDNESS	$\mathcal{U}$, DEFINEDNESS	MUST
ACETO-HENNESSY	ACETO-HENNESSY	MUST
Canonical NDBPs for E-errors (Def. 3.4 with $\|E\| \geqslant 2$		
INCONSISTENT	INCONSISTENT	INCONSISTENT
EQUALITY	EQUALITY	EQUALITY
INCLUSION	INCLUSION	INCLUSION
REFINEMENT	REFINEMENT	REFINEMENT
$\lesssim^{\mathsf{ND}(\emptyset)}$	$\lesssim^{\mathsf{ND}(\emptyset)}$	INCONSISTENT
$\lesssim^{\mathsf{ND}(E)}$	$\lesssim^{\mathsf{ND}(E)}$	INCONSISTENT
$\lesssim^{\mathsf{ND}(C)}$ $(C \neq \emptyset, E)$	$\lesssim^{\mathsf{ND}(C)}$	$\lesssim^{\mathsf{ND}(C)}$
$\equiv^{\mathsf{ND}(\emptyset)}$	$\equiv^{\mathsf{ND}(\emptyset)}$	INCONSISTENT
$\equiv^{\mathsf{ND}(E)}$	$\equiv^{\mathsf{ND}(E)}$	INCONSISTENT
$\equiv^{\mathsf{ND}(C)}$ $(C \neq \emptyset, E)$	$\equiv^{\mathsf{ND}(C)}$	$\equiv^{\mathsf{ND}(C)}$

Figure 10: 1-testing and 0-testing all the NDBPs in this paper

(3) *Let E be a nonempty finite set. Every inflationary NDBP, deflationary NDBP and NDBC for E-errors is 1-testable.*

(4) *Let E be a finite set, and $\sqsubseteq$ a consistent NDBP for E-errors with a neutral element x. Then x has the form $\emptyset + D$ with every $e \in D$ neutral, and $\sqsubseteq$ is 1-testable.*

(5) *Let E be a finite set, and $\sqsubseteq$ an inflationary NDBP for E-errors. Then $\sqsubseteq$ is 0-testable iff it is of the form $\bigcap_{C \in \mathcal{C}} \lesssim^{\mathsf{ND}(C)}$ for $\mathcal{C} \subseteq (\mathcal{P}E) \setminus \{\emptyset, E\}$.*

(6) *Let E be a finite set, and $\equiv$ a NDBC for E-errors. Then $\equiv$ is 0-testable iff it is of the form $\bigcap_{C \in \mathcal{C}} \equiv^{\mathsf{ND}(C)}$ for $\mathcal{C} \subseteq (\mathcal{P}E) \setminus \{\emptyset, E\}$.*

Proof. (1) From Prop. 3.2(1).

(2) Such a DBC must be of the form $\bigcap_{C \in \mathcal{C}} \equiv^{(C)}$, for some $\mathcal{C} \subseteq \mathcal{P}E \setminus \{\emptyset, E\}$.

(3) From Prop. 3.5.

(4) In any semilattice, if $u \mathsf{ or } v$ is neutral then u (and likewise v) are neutral, because for any w

$$w \mathsf{ or } u = (w \mathsf{ or } u) \mathsf{ or } (u \mathsf{ or } v) = w \mathsf{ or } (u \mathsf{ or } v) = w$$

Since $\mathsf{inl}\ b$ cannot be neutral for a consistent NDBP, x must be $\emptyset + D$ with every $e \in E$ neutral.

Suppose $K + C, K' + C' \in \mathcal{P}^{>0}(\mathbb{B} + E)$ are such that $K + C \not\sqsubseteq K' + C'$. We wish to find $f \in \mathcal{P}^{>0}(1 + E)^{\mathbb{B}}$ such that

$$P_f(K + C) \not\sqsubseteq_1 P_f(K' + C') \tag{3.1}$$

where $P_f : y \mapsto x \cup \mathsf{NDif}^E_1(y, f)$. Since $K + C \not\sqsubseteq_{\mathbb{B}} K' + C'$, one of the following cases must apply.

- If $\{\mathsf{t}\} + C \not\sqsubseteq \{\mathsf{t}\} + C'$, let f be the constant function to $\{\mathsf{inl}\ *\}$.
- If $\{\mathsf{t}, \mathsf{f}\} + C \not\sqsubseteq \{\mathsf{t}\} + C'$ and $a \in K \setminus K'$, let f be $\{a.\{\mathsf{inl}\ *\}, i \neq a.x\}$.
- Likewise if $\{\mathsf{t}\} + C' \not\sqsubseteq \{\mathsf{t}, \mathsf{f}\} + C'$ and $a \in K' \setminus K$.

(5) From Prop. 3.5.

(6) From Prop. 3.5.

4. Relational Lifting

4.1. **Lifting a Relation.** In Sect. 2.3 we saw how a boolean precongruence gives an ordering at every ground type. We generalize this to a way of lifting relations.

Definition 4.1.

(1) Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. For any sets A, B and relation $A \xrightarrow{\mathcal{R}} B$, we define a relation $A + E \xrightarrow{\sqsubseteq_{\mathcal{R}}} B + E$ as follows.
   (a) $\mathsf{inl}\ a \sqsubseteq_{\mathcal{R}} \mathsf{inl}\ b$ when $a\ \mathcal{R}\ b$ or $\mathsf{inl}\ \mathsf{t} \sqsubseteq \mathsf{inl}\ \mathsf{f}$.
   (b) $\mathsf{inr}\ e \sqsubseteq_{\mathcal{R}} \mathsf{inl}\ b$ when $\mathsf{inr}\ e \sqsubseteq \mathsf{inl}\ \mathsf{t}$.
   (c) $\mathsf{inl}\ a \sqsubseteq_{\mathcal{R}} \mathsf{inr}\ e$ when $\mathsf{inl}\ \mathsf{t} \sqsubseteq \mathsf{inr}\ e$.
   (d) $\mathsf{inr}\ e \sqsubseteq_{\mathcal{R}} \mathsf{inr}\ e'$ when $\mathsf{inr}\ e \sqsubseteq \mathsf{inr}\ e'$.

(2) Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. For any sets A, B and relation $A \xrightarrow{\mathcal{R}} B$, we define a relation $\mathcal{P}(A + E) \xrightarrow{\sqsubseteq_{\mathcal{R}}} \mathcal{P}(B + E)$ setting $K + D \sqsubseteq_{\mathcal{R}} K' + D'$ when all the following conditions are met.
   (a) $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$.
   (b) if $\{\mathsf{t}, \mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$ then for every $a \in K$ there exists $b \in K'$ such that $a\ \mathcal{R}\ b$.
   (c) if $\{\mathsf{t}\} + D \not\sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ then for every $b \in K'$ there exists $a \in K$ such that $a\ \mathcal{R}\ b$.

This construction has the "stable relator" properties stipulated in [Bal00, HT00, HJ04]. We write ";" for relational composition in diagrammatic order.

Lemma 4.2.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. Write $T : Z \mapsto Z + E$*
   (a) *For sets A, B and relations $A \xrightarrow{\mathcal{R}, \mathcal{S}} B$, if $\mathcal{R} \subseteq \mathcal{S}$ then $\sqsubseteq_{\mathcal{R}} \subseteq \sqsubseteq_{\mathcal{S}}$.*
   (b) *For a set A, we have $\mathsf{id}_{TA} \subseteq \sqsubseteq_{\mathsf{id}_A} = \sqsubseteq_A$.*

(c) *For sets* A, B, C *and relations* $A \xrightarrow{\mathcal{R}} B \xrightarrow{\mathcal{S}} C$ *we have*

$$\sqsubseteq_{\mathcal{R}} ; \sqsubseteq_{\mathcal{S}} \subseteq \sqsubseteq_{\mathcal{R};\mathcal{S}} \tag{4.1}$$

If $\sqsubseteq$ *is consistent, then (4.1) is an equality.*

(d) *For sets* A, B, A', B'*, a relation* $A \xrightarrow{\mathcal{R}} B$ *and functions* $A' \xrightarrow{f} A$ *and* $B' \xrightarrow{g} B$ *we have*

$$\sqsubseteq_{(f\times g)^{-1}\mathcal{R}} = (Tf \times Tg)^{-1} \sqsubseteq_{\mathcal{R}}$$

(2) *Let* E *be a finite set, and let* $\sqsubseteq$ *be a NDBP for* E*-errors. Write* $T : Z \mapsto \mathcal{P}(Z + E)$*.*

(a) *For sets* A, B *and relations* $A \xrightarrow{\mathcal{R},\mathcal{S}} B$ *, if* $\mathcal{R} \subseteq \mathcal{S}$ *then* $\sqsubseteq_{\mathcal{R}} \subseteq \sqsubseteq_{\mathcal{S}}$*.*

(b) *For a set* A*, we have* $\mathsf{id}_{TA} \subseteq \sqsubseteq_{\mathsf{id}_A} = \sqsubseteq_A$

(c) *For sets* A, B, C *and relations* $A \xrightarrow{\mathcal{R}} B \xrightarrow{\mathcal{S}} C$ *we have*

$$\sqsubseteq_{\mathcal{R}} ; \sqsubseteq_{\mathcal{S}} \subseteq \sqsubseteq_{\mathcal{R};\mathcal{S}} \tag{4.2}$$

If $\sqsubseteq$ *is consistent, then (4.2) is an equality, and this remains true if we restrict* $\mathcal{P}$ *to nonempty, countable or finite subsets.*

(d) *For sets* A, B, A', B'*, a relation* $A \xrightarrow{\mathcal{R}} B$ *and functions* $A' \xrightarrow{f} A$ *and* $B' \xrightarrow{g} B$ *we have*

$$\sqsubseteq_{(f\times g)^{-1}\mathcal{R}} = (Tf \times Tg)^{-1} \sqsubseteq_{\mathcal{R}}$$

Proof.

(1) (a) Trivial.
(b) Trivial.
(c) If $\mathsf{inl}\ a \sqsubseteq_{\mathcal{R}} \mathsf{inr}\ e \sqsubseteq_{\mathcal{S}} \mathsf{inl}\ b$, then $\mathsf{inl}\ \mathsf{t} \sqsubseteq \mathsf{inr}\ e \sqsubseteq \mathsf{inl}\ \mathsf{f}$ and so making $\sqsubseteq$ inconsistent. The other seven cases are trivial, and so is the reverse inclusion assuming $\sqsubseteq$ consistent.
(d) Trivial.

(2) (a) Trivial.
(b) Trivial.
(c) Suppose $K + D \sqsubseteq_{\mathcal{R}} K' + D' \sqsubseteq_{\mathcal{S}} K'' + D''$. Then

$$\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D' \sqsubseteq \{\mathsf{t}\} + D''$$

Suppose $\{\mathsf{t}, \mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + D''$, and $a \in K$. If $\{\mathsf{t}, \mathsf{f}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ then

$$\{\mathsf{t}, \mathsf{f}\} + D \sqsubseteq \{\mathsf{t}\} + D' \sqsubseteq \{\mathsf{t}\} + D''$$

a contradiction, so there is $b \in K'$ such that $a\ \mathcal{R}\ b$. If $\{\mathsf{t}, \mathsf{f}\} + D' \sqsubseteq \{\mathsf{t}\} + D''$ then

$$\{\mathsf{t}, \mathsf{f}\} + D \sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D' \sqsubseteq \{\mathsf{t}\} + D''$$

a contradiction, so there is $c \in K''$ such that $b\ \mathcal{S}\ c$. Dually for requirement (2c) of Def. 4.1.

Conversely, suppose $\sqsubseteq$ is consistent, and $x \sqsubseteq_{\mathcal{R};\mathcal{S}} z$. We shall show that there is $y \in \mathcal{P}(B + E)$ such that

- $x \sqsubseteq_{\mathcal{R}} y$ and $y \sqsubseteq_{\mathcal{S}} z$
- if x and z are both nonempty (resp. finite, countable) then so is y.

We write $x = K + D$ and $z = K'' + D''$. Define $L \subseteq B$ as follows: if $\{\mathsf{t},\mathsf{f}\} + D \sqsubseteq \{\mathsf{t}\} + D''$ then $L \stackrel{\text{def}}{=} \emptyset$, otherwise for each $a \in K$ we pick $f(a) \in B$ and $g(a) \in C$ such that $a\ \mathcal{R}\ f(a)$ and $f(a)\ \mathcal{S}\ g(a)$, and then we set L to be the range of $K \xrightarrow{f} B$. We define $L'' \subseteq B$ the same way in the opposite direction, and set $y \stackrel{\text{def}}{=} (L \cup L'') + (D \cup D'')$.
We note that for each $b \in L \cup L'$ there is $a \in A$ such that $a\ \mathcal{R}\ b$ and $c \in C$ such that $b\ \mathcal{S}\ c$.
We show $x \sqsubseteq_{\mathcal{R}} y$ as follows. For condition (2a),

$$\begin{array}{rcl} \{\mathsf{t}\} + D & = & (\{\mathsf{t}\} + D) \cup (\{\mathsf{t}\} + D) \\ & \sqsubseteq & (\{\mathsf{t}\} + D) \cup (\{\mathsf{t}\} + D'') \\ & = & \{\mathsf{t}\} + (D \cup D'') \end{array}$$

For condition (2b), suppose $\{\mathsf{t},\mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + (D \cup D'')$. Then $\{\mathsf{t},\mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + D''$, for otherwise

$$\begin{array}{rcl} \{\mathsf{t},\mathsf{f}\} + D & = & (\{\mathsf{t}\} + D) \cup (\{\mathsf{t},\mathsf{f}\} + D) \\ & \sqsubseteq & (\{\mathsf{t}\} + D) \cup (\{\mathsf{t}\} + D'') \\ & = & \{\mathsf{t}\} + (D \cup D'') \end{array}$$

contradicting our assumption. Therefore for each $a \in K$ we have $f(a) \in L$ and $a\ \mathcal{R}\ f(a)$. For condition (2c) of Def. 4.1, we observe that for each $b \in L \cup L'$ there is $a \in A$ such that $a\ \mathcal{R}\ b$.
The proof of $y \sqsubseteq_{\mathcal{S}} z$ is similar.
Clearly if both x and z are finite (resp. countable) then so is y. Suppose both x and z are nonempty but y is empty. Then D and D'' are empty. Since D is empty, K must be nonempty, but L is empty so we must have $\{\mathsf{t},\mathsf{f}\} + \emptyset \sqsubseteq \{\mathsf{t}\} + \emptyset$. By the dual argument $\{\mathsf{t}\} + \emptyset \sqsubseteq \{\mathsf{t},\mathsf{f}\} + \emptyset$, contradicting our consistency assumption by Lemma 2.7.

(d) Trivial.

Following [HT00], we can express the general construction $\mathcal{R} \mapsto \sqsubseteq_{\mathcal{R}}$ in terms of the special case $A \mapsto \sqsubseteq_A$.

Proposition 4.3. *Let A and B be sets and $A \xrightarrow{\mathcal{R}} B$ a relation. We write*

$$\mathrm{graph}(\mathcal{R}) \stackrel{\text{def}}{=} \{(x,y) \in A \times B \mid x\ \mathcal{R}\ y\}$$

$$\begin{tikzcd} & \mathrm{graph}(\mathcal{R}) \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ A & & B \end{tikzcd}$$

for the two projections from the graph of $\mathcal{R}$.

(1) *Let E be a set, and $\sqsubseteq$ a consistent DBP for E-errors. Write $T : Z \mapsto Z + E$. Then $\sqsubseteq_{\mathcal{R}}$ is the composite*

$$TA \xrightarrow{(TA \times T\pi)^{-1} \sqsubseteq_A} T\mathrm{graph}(\mathcal{R}) \xrightarrow{(T\pi' \times TB)^{-1} \sqsubseteq_B} TB \tag{4.3}$$

(2) *Let E be a finite set, and $\sqsubseteq$ a consistent NDBP for E-errors. Write $T : Z \mapsto \mathcal{P}(Z + E)$. Then $\sqsubseteq_{\mathcal{R}}$ is the composite (4.3), and this remains true if we restrict $\mathcal{P}$ to nonempty, finite or countable sets.*

Proof. The relation $\mathcal{R}$ is the composite

$$A \xrightarrow{(A\times\pi)^{-1}\mathsf{id}_A} \mathrm{graph}(\mathcal{R}) \xrightarrow{(\pi'\times B)^{-1}\mathsf{id}_B} B$$

So Lemma 4.2 gives us

$$\begin{aligned} \sqsubseteq_{\mathcal{R}} &= \sqsubseteq_{(A\times\pi)^{-1}\mathsf{id}_A} ; \sqsubseteq_{(\pi'\times B)^{-1}\mathsf{id}_B} \\ &= (TA \times T\pi)^{-1} \sqsubseteq_{\mathsf{id}_A} ; (T\pi' \times TB)^{-1} \sqsubseteq_{\mathsf{id}_B} \\ &= (TA \times T\pi)^{-1} \sqsubseteq_A ; (T\pi' \times TB)^{-1} \sqsubseteq_B \end{aligned}$$

□

Finally we see that our construction is compatible with union and amb.

Lemma 4.4.

(1) *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. If $A \xrightarrow{\mathcal{R}} B$ a relation, then $\sqsubseteq_{\mathcal{R}}$ is preserved by I-ary union, for each set I.*

(2) *Let E be a finite set, and let $\sqsubseteq$ be an ABP for E-errors. If $A \xrightarrow{\mathcal{R}} B$ a relation, then $\sqsubseteq_{\mathcal{R}}$ is preserved by I-ary amb, for each set I.*

Proof. (1) Given a set I, and for each $i \in I$ elements $K_i + D_i \in \mathcal{P}(A+E)$ and $K'_i + D'_i \in \mathcal{P}(B+E)$ such that $K_i + D_i \sqsubseteq_{\mathcal{R}} K'_i + D'_i$, we want to show

$$\bigcup_{i\in I} K_i + \bigcup_{i\in I} D_i \sqsubseteq_{\mathcal{R}} \bigcup_{i\in I} K'_i + \bigcup_{i\in I} D'_i \tag{4.4}$$

We have $\{\mathsf{t}\} + D_i \sqsubseteq \{\mathsf{t}\} + D'_i$ for each $i \in I$, and so

$$(\{\mathsf{t}\} + \emptyset) \cup \bigcup_{i\in I}(\{\mathsf{t}\} + D_i) \quad \sqsubseteq \quad (\{\mathsf{t}\} + \emptyset) \cup \bigcup_{i\in I}(\{\mathsf{t}\} + D'_i) \tag{4.5}$$

using the fact that $\mathcal{P}^{>0}(\mathbb{B} + E)$ is finite, so $1 + I$-ary union reduces to a finite nonempty union and is therefore monotone wrt $\sqsubseteq$. (4.5) reduces to

$$\{\mathsf{t}\} + \bigcup_{i\in I} D_i \quad \sqsubseteq \quad \{\mathsf{t}\} + \bigcup_{i\in I} D'_i \tag{4.6}$$

Suppose

$$\{\mathsf{t},\mathsf{f}\} + \bigcup_{i\in I} D_i \not\sqsubseteq \{\mathsf{t}\} + \bigcup_{i\in I} D'_i \tag{4.7}$$

and $a \in \bigcup_{i\in I} K_i$. Pick $\hat{\imath} \in I$ such that $a \in K_{\hat{\imath}}$. If $\{\mathsf{t},\mathsf{f}\} + D_{\hat{\imath}} \sqsubseteq \{\mathsf{t}\} + D'_{\hat{\imath}}$, then the union of this with (4.6) contradicts (4.7). So this is not the case, and there is $b \in K_{\hat{\imath}}$ such that $a \mathrel{\mathcal{R}} b$. Dually for requirement (2c). □

(2) The same argument, replacing $\cup$ by amb and $\emptyset$ by $\{\mathsf{d}\}$. □

4.2. **Lifting a Preorder.** We turn to the case in which the relation being lifted is a preorder.

Proposition 4.5.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. Let $(A, \leqslant)$ be a preordered set. Then*
   (a) $(A + E, \sqsubseteq_{\leqslant}, \mathsf{inr})$ *is a preordered set with E-errors, compatible with $\sqsubseteq$*
   (b) *the following is monotone:*

$$\begin{array}{ccc} (A, \leqslant) & \longrightarrow & (A + E, \sqsubseteq_{\leqslant}) \\ x & \mapsto & \mathsf{inl}\ x \end{array}$$

(2) *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. Let $(A, \leqslant)$ be a preordered set. Then*
   (a) $(\mathcal{P}(A + E), \sqsubseteq_{\leqslant}, \cup, \mathsf{inr})$ *is a semilattice with E-errors, compatible with $\sqsubseteq$, with a join operation $\bigcup_I$ for every set I*
   (b) *the following is monotone:*

$$\begin{array}{ccc} (A, \leqslant) & \longrightarrow & (\mathcal{P}(A + E), \sqsubseteq_{\leqslant}) \\ x & \mapsto & \{\mathsf{inl}\ x\} \end{array}$$

Proof.

(1) $\sqsubseteq_{\leqslant}$ is a preorder by Lemma 4.2(4.2)(1b–1c). Compatibility is straightforward case analysis.
(2) $\sqsubseteq_{\leqslant}$ is a preorder by Lemma 4.2(2)(2b–2c). Compatibility is proved as follows:
   (a) Suppose $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ Then for $K + C \in \mathcal{P}(B + E)$ we have $\{\mathsf{t}\} + (C \cup D) \sqsubseteq \{\mathsf{t}\} + (C \cup D')$ and hence

$$K + (C \cup D) \sqsubseteq_{\leqslant} K + (C \cup D')$$

   (b) Suppose $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ Then for $K + C, K' + C' \in \mathcal{P}(B + E)$ we have $\{\mathsf{t}\} + (C \cup D) \sqsubseteq \{\mathsf{t}, \mathsf{f}\} + (C \cup C' \cup D)$ and hence

$$K + (C \cup D) \sqsubseteq_{\leqslant} (K \cup K') + (C \cup C' \cup D)$$

   (c) Dually.

   It is clear that $\bigcup_I$ is a join operation, and its monotonicity is given by Lemma 4.4(1). $\square$

The following notation allows us to express our results in their full generality.

Definition 4.6. Let $\underline{B}$ be a preordered semilattice and let A be a set. Then $\mathcal{P}^{\underline{B}} A$ is the set of all $x \in A$ such that $\underline{B}$ has an x-ary join operation.

Proposition 4.7.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. Let $(A, \leqslant)$ be a preordered set, and let $\underline{B} = (X, \leqslant, \mathsf{raise})$ be a preordered set with E-errors compatible with $\sqsubseteq$. Then the following function is monotone:*

$$\begin{array}{ccccc} (A + E, \sqsubseteq_{\leqslant}) & \times & \underline{B}^{(A,\leqslant)} & \longrightarrow & \underline{B} \\ x, & & f & \mapsto & \mathsf{Dcase}^{E}_{\underline{B},(}(x, f) \end{array}$$

(2) *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. Let $(A,\leqslant)$ be a preordered set, and let $\underline{B} = (X,\leqslant,\mathsf{or},\mathsf{raise})$ be a preordered semilattice with E-errors compatible with $\sqsubseteq$. For any set C, we write*

$$\mathcal{P}^{\underline{B}}C \quad \stackrel{\text{def}}{=} \quad \{R \subseteq C \mid \underline{B} \textit{ has a } R\textit{-ary join operation}\}$$

Then the following (total) function is monotone:

$$\begin{array}{ccccc} (\mathcal{P}^{\underline{B}}(A+E),\sqsubseteq_{\leqslant}) & \times & \underline{B}^{(A,\leqslant)} & \longrightarrow & \underline{B} \\ x, & & f & \mapsto & \mathsf{NDcase}^{E}_{A,\underline{B}}(x,f) \end{array}$$

In both parts $\underline{B}^{(A,\leqslant)}$ means the preordered set of monotone functions from $(A,\leqslant)$ to $\underline{B}$.

Proof.

(1) Write $\underline{B} = (B,\leqslant',\mathsf{raise})$. Suppose we are given $x \sqsubseteq_{\leqslant} y \in A+E$ and monotone $g \xrightarrow{f} (A,\leqslant)\ (B,\leqslant')$ such that $f \leqslant' g$. We need to show

$$\mathsf{Dcase}^{E}_{A,\underline{B}}(x,f) \leqslant' \mathsf{Dcase}^{E}_{A,\underline{B}}(y,g)$$

- If $x = \mathsf{inl}\ a$ and $y = \mathsf{inl}\ a'$, then either $a \leqslant a'$, so LHS $= fa \leqslant' ga \leqslant' ga' =$ RHS, or else $\sqsubseteq$ is inconsistent, giving LHS $\leqslant'$ RHS immediately.
- If $x = \mathsf{inr}\ e$ and $y = \mathsf{inl}\ a$, then $\mathsf{inr}\ e \sqsubseteq \mathsf{inl}\ \mathsf{true}$, so LHS $= \mathsf{raise}\ e \leqslant' ga' =$ RHS.
- Dually if $x = \mathsf{inl}\ a$ and $y = \mathsf{inr}\ e$.
- If $x = \mathsf{inr}\ e$ and $y = \mathsf{inr}\ e$, then $\mathsf{inl}\ e \sqsubseteq \mathsf{inr}\ e$, so LHS $= \mathsf{raise}\ e \leqslant' \mathsf{raise}\ e'$.

(2) Write $\underline{B} = (B,\leqslant',\mathsf{or},\mathsf{raise})$. Suppose $K+D, K'+D' \in \mathcal{P}^{\underline{B}}(A+E)$ are such that $K+D \sqsubseteq_{\leqslant} K'+D'$, and monotone functions $(A,\leqslant) \xrightarrow{f,g} (B,\leqslant')$ are such that $fa \leqslant' ga$ for all $a \in A$. We show

$$\begin{aligned} \mathsf{NDcase}^{E}_{A,\underline{B}}(K+D,f) &\leqslant \mathsf{NDcase}^{E}_{A,\underline{B}}(K+D,f)\ \mathsf{or}\ \mathsf{NDcase}^{E}_{A,\underline{B}}(K'+D',g) && (4.8)\\ &\leqslant \mathsf{NDcase}^{E}_{A,\underline{B}}(K'+D',g) && (4.9) \end{aligned}$$

We prove just (4.8), as (4.9) is proved dually.

If $\{\mathsf{t}\}+D \sqsubseteq \{\mathsf{t},\mathsf{f}\}+D'$, then we have

$$\begin{array}{ll} & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk\\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \\ = & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk\\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{orraise}\ D \\ & \text{(by absorption)} \\ \leqslant' & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk\\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{or}\ \mathsf{choose}_{K'+D'}\left\{\begin{array}{ll}\mathsf{inl}\ a. & gk\\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{orraise}\ D' \\ & \text{(since } \{\mathsf{t}\}+D \sqsubseteq \{\mathsf{t},\mathsf{f}\}+D'\text{)} \\ = & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk\\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{or}\ \mathsf{choose}_{K'+D'}\left\{\begin{array}{ll}\mathsf{inl}\ a. & gk\\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \\ & \text{(by absorption)} \end{array}$$

On the other hand, if $\{\mathsf{t}\}+D \not\sqsubseteq \{\mathsf{t},\mathsf{f}\}+D'$ then (by the Axiom of Choice) there is a function $K' \xrightarrow{h} K$ such that $ha \leqslant' a$, and hence $f(ha) \leqslant' ga$, for each $a \in K$.

We then have

$$\begin{array}{ll} & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk \\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \\ = & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk \\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{orraise}\ D \\ & \text{(by (2))} \\ \leqslant' & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk \\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{orraise}\ D' \\ & \text{(since } \{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'\text{)} \\ = & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk \\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{or}\ \mathsf{choose}_{K'+D'}\left\{\begin{array}{ll}\mathsf{inl}\ a. & f(ha) \\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \\ & \text{(supremum characterization of } \mathsf{choose}\text{)} \\ \leqslant' & \mathsf{choose}_{K+D}\left\{\begin{array}{ll}\mathsf{inl}\ a. & fk \\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \quad \mathsf{or}\ \mathsf{choose}_{K'+D'}\left\{\begin{array}{ll}\mathsf{inl}\ a. & ga \\ \mathsf{inr}\ e. & \mathsf{raise}\ e\end{array}\right. \\ & \text{(monotonicity of } \mathsf{or} \text{ and } \mathsf{choose}\text{)} \end{array}$$

□

Remark 4.8. If E is a countable set, Prop. 4.7(2) remains true provided that $\underline{B}$ is a preordered ω-semilattice with E-errors.

We now give the proof of Prop. 2.16.

(1) (a) From Prop. 4.5(1a).
   (b) By Prop. 4.5(1a), $(B+E, \sqsubseteq_B, \mathsf{inr}\ )$ is compatible with $\sqsubseteq$, and Prop. 4.7(1) then gives monotonicity.
   (c) $\sqsubseteq$ meets the hypotheses of Prop. 2.19(1) and therefore must be $\sqsubseteq_{\mathbb{B}}$.
   (d) Special case of Lemma 4.2(1d).

(2) (a) From Prop. 4.5(2a).
   (b) By Lemma 4.4(1).
   (c) By Prop. 4.5(2a), $(\mathcal{P}(A+E), \sqsubseteq_{\leqslant}, \cup, \mathsf{inr})$ is compatible with $\sqsubseteq$, and $\bigcup_I$ is a join operation for every set I. So Prop. 4.7(2) gives monotonicity.
   (d) $\sqsubseteq$ meets the hypotheses of Prop. 2.19(2), so $\sqsubseteq$ is the restriction of $\sqsubseteq_{\mathbb{B}}$ to $\mathcal{P}^{>0}(\mathbb{B} + E)$. For uniqueness, suppose $\leqslant$ is a NDBP with empty set for E-errors that extends $\sqsubseteq$. Then $\leqslant$ meets the hypotheses of Prop. 2.19(2), and therefore must be $\sqsubseteq_{\mathbb{B}}$. □
   (e) Special case of Lemma 4.2(2d).

Putting together our results, we shall see that relational lifting gives a free construction on a poset.

Proposition 4.9.

(1) *Let E be a set, and let $\sqsubseteq$ be a DBP for E-errors. Write* $\mathbf{Poset}^{\sqsubseteq}$ *for the* **Poset***-enriched category of posets with E-errors, compatible with $\sqsubseteq$. Then the forgetful* **Poset***-enriched functor* $\mathbf{Poset}^{\sqsubseteq} \xrightarrow{U^{\sqsubseteq}} \mathbf{Poset}$ *is monadic, with $(A, \leqslant)$ mapped by the left adjoint to* $\mathsf{quotient}(A + E, \sqsubseteq_{\leqslant})$ *with raise operation* $e \mapsto [\mathsf{inr}\ e]$ *and unit* $a \mapsto [\mathsf{inl}\ a]$.

(2) *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors.*

(a) *Write* $\mathbf{Poset}^{\sqsubseteq}$ *for the* $\mathbf{Poset}$*-enriched category of partially ordered semilattices with* E*-errors, compatible with* $\sqsubseteq$*. Then the forgetful* $\mathbf{Poset}$*-enriched functor* $\mathbf{Poset}^{\sqsubseteq} \xrightarrow{U^{\sqsubseteq}} \mathbf{Poset}$ *is monadic, with* $(A, \leqslant)$ *mapped by the left adjoint to* $\mathsf{quotient}(\mathcal{P}^{>0,<\aleph_0}(A+E), \sqsubseteq_{\leqslant})$ *with semilattice operation* $[U],[V] \mapsto [U \cup V]$*, raise operation* $e \mapsto [\{\mathsf{inr}\ e\}]$ *and unit* $a \mapsto [\{\mathsf{inl}\ a\}]$*.*
(b) *Similarly for partially ordered* ω*-semilattices, with the left adjoint mapping* $(A, \leqslant)$ *to* $\mathsf{quotient}(\mathcal{P}^{>0,\leqslant\aleph_0}(A+E), \sqsubseteq_{\leqslant})$*.*

Proof.

(1) Prop. 4.5(1) gives an unenriched left adjoint. Thus we have a bijection

$$\theta \ : \ \mathbf{Poset}((A,\leqslant),B) \ \cong \ \mathbf{Poset}^{\sqsubseteq}((A+E,\mathsf{inr}),(B,\mathsf{raise})) \tag{4.10}$$

whose inverse is the monotone function $f \mapsto \mathsf{inl}; f$. Prop. 4.7(1) tells us that θ is monotone, hence an order isomorphism, giving a $\mathbf{Poset}$-enriched left adjoint to $U^{\sqsubseteq}$.

Let K be the unique $\mathbf{Poset}$-enriched comparison functor from $\mathbf{Poset}^{\sqsubseteq}$ to the enriched category of algebras. Beck's theorem tells us that K is invertible as an unenriched functor. K is locally order-reflecting because $U^{\sqsubseteq}$ is, so its inverse is a $\mathbf{Poset}$-enriched functor.

(2) Similar. □

4.3. **Simulation.** In [Las98], a nondeterministic language with divergence is studied, and various notions of simulation for a language are defined, named "lower", "upper" and "convex". We now see how they can be systematically defined from a CBP. We recall that an *labelled transition system* (LTS) over a set Act of *actions*[2] is a coalgebra for the endofunctor $\mathcal{P}(\mathsf{Act} \times -)$ on $\mathbf{Set}$.

Definition 4.10. Let E be a finite set of errors, and let Act be a set of actions.

- A *LTS with* E*-errors* over Act is a coalgebra (X, ξ) for the endofunctor $\mathcal{P}(\mathcal{A} \times - + E)$ on $\mathbf{Set}$. For convenience, we represent ξ as a relation $\rightarrow \subseteq X \times \mathsf{Act} \times X$ and a relation $\lightning \subseteq X \times E$. For $x \in X$ we write

$$\mathsf{Errors}(x) \stackrel{\text{def}}{=} \{e \in E \mid x \lightning e\}$$

- Such a system is *lively* (resp. *deterministic*, *countably branching*) when for each $x \in X$ the set $\xi(x)$ is nonempty (resp. singleton, countable).

Definition 4.11. Let E be a finite set and let $\sqsubseteq$ be a NDBP for E-errors. Let Act be a set, and let $\underline{X} = (X, \xi)$ and $\underline{X}' = (X', \xi')$ be LTSs with E-errors over Act.

(1) A $\sqsubseteq$ *simulation* from $\underline{X}$ to $\underline{X}'$ is a relation $X \xrightarrow{\mathcal{R}} X'$ such that if $x\ \mathcal{R}\ x'$ then $\xi(x) \ \sqsubseteq_{\mathcal{R}} \ \xi'(x')$ i.e.
   - $\{\mathsf{t}\} + \mathsf{Errors}(x) \sqsubseteq \{\mathsf{t}\} + \mathsf{Errors}(x')$
   - if $\{\mathsf{t},\mathsf{f}\} + \mathsf{Errors}(x) \not\sqsubseteq \{\mathsf{t}\} + \mathsf{Errors}(x')$ then $x \xrightarrow{a} y$ implies that there is y' such that $x' \xrightarrow{a} y'$ and $y\ \mathcal{R}\ y'$

[2] More generally, we can consider LTSs over a *game graph* of actions [LL09]. All of our theory works in that more general situation.

- if $\{\mathsf{t}\} + \mathsf{Errors}(x) \not\sqsubseteq \{\mathsf{t},\mathsf{f}\} + \mathsf{Errors}(x')$ then $x' \xrightarrow{a} y'$ implies that there is y such that $x \xrightarrow{a} y$ and $y \mathrel{\mathcal{R}} y'$.

(2) The greatest simulation from $\underline{X}$ to $\underline{X}'$ is called $\sqsubseteq$ *similarity*.

Lemma 4.2(2) tells us that $\sqsubseteq$ similarity is a preorder on nodes of LTSs with E-errors over Act. We write $\lesssim^{\sqsubseteq}_{\mathsf{Act}}$ for this preorder.

We consider some particular cases of consistent NDBPs (since *any* relation between transition systems is an INCONSISTENT simulation). Firstly, transition systems without no errors. We see that

- an INCLUSION simulation is just a simulation
- a REFINEMENT simulation is the converse of a simulation
- an EQUALITY simulation is a bisimulation.

Next we consider transition systems with divergence, writing $x \Uparrow$ to mean that x may diverge. A relation $\mathcal{R}$ between transition systems with divergence is

- a LOWER simulation [How89, Las98, Mor98, Pit01, Uli92] when for $x \mathrel{\mathcal{R}} x'$, if $x \xrightarrow{a} y$ then there is y' such that $y \mathrel{\mathcal{R}} y'$ and $x' \xrightarrow{a} y'$
- an UPPER simulation [Las98, Mor98, Pit01, Uli92] when for $x \mathrel{\mathcal{R}} x'$, if $x \not\Uparrow$ then
  - $y \not\Uparrow$
  - if $x' \xrightarrow{a} y'$ then there exists y such that $y \mathrel{\mathcal{R}} y'$ and $x \xrightarrow{a} y$
- a SMASH simulation when for $x \mathrel{\mathcal{R}} x'$, if $x \not\Uparrow$ then
  - $y \not\Uparrow$
  - if $x \xrightarrow{a} y$ then there exists y' such that $y \mathrel{\mathcal{R}} y'$ and $x' \xrightarrow{a} y'$
  - if $x' \xrightarrow{a} y'$ then there exists y such that $y \mathrel{\mathcal{R}} y'$ and $x \xrightarrow{a} y$
- an INCLUSION simulation when for $x \mathrel{\mathcal{R}} x'$,
  - if $x \xrightarrow{a} y$ then there is y' such that $y \mathrel{\mathcal{R}} y'$ and $x' \xrightarrow{a} y'$
  - if $x \Uparrow$ then $x' \Uparrow$.

  equivalently when it is a LOWER simulation and its converse is an UPPER simulation
- a CONVEX simulation [Las98, Mor98, Ong93, Pit01] (or *pre-bisimulation* [HP80, Mil81, Wal90] or *partial bisimulation* [Abr91]) when it is both a LOWER simulation and an UPPER simulation
- a REFINEMENT simulation [How96, Las98, Mor98, Pit01] when its converse is an INCLUSION simulation
- a LOWER CONGRUENCE simulation or *lower bisimulation* [How89, Las98, Mor98, Pit01] when $\mathcal{R}$ and its converse are LOWER simulations
- an UPPER CONGRUENCE simulation or *upper bisimulation* [Las98, Mor98, Pit01] when $\mathcal{R}$ and its converse are UPPER simulations
- an EQUALITY simulation or *convex bisimulation* [Las98, Mor98, Pit01] or *refinement bisimulation* [How96, Las98, Mor98, Pit01] when for $x \mathrel{\mathcal{R}} x'$,
  - if $x \xrightarrow{a} y$ then there is y' such that $y \mathrel{\mathcal{R}} y'$ and $x' \xrightarrow{a} y'$
  - if $x' \xrightarrow{a} y'$ then there is y such that $y \mathrel{\mathcal{R}} y'$ and $x \xrightarrow{a} y$
  - $x \Uparrow$ iff $x' \Uparrow$

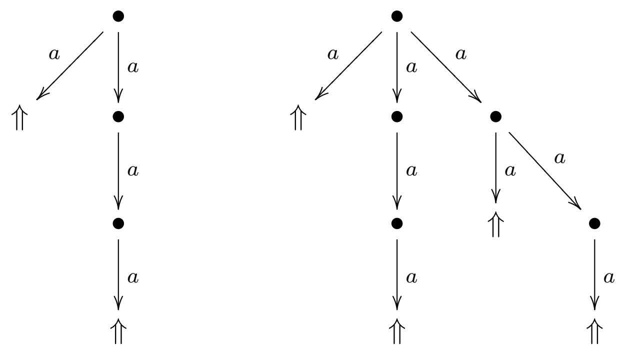

Figure 11: Mutually convex similar, but neither lower nor upper bisimilar

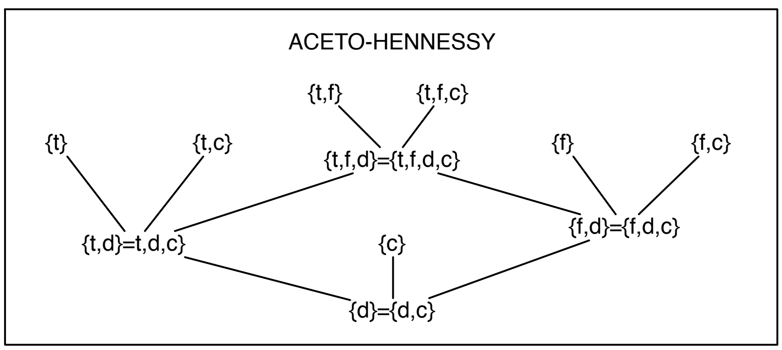

Figure 12: NDBP for divergence (d) and deadlock (c) used in [AH92]

equivalently when $\mathcal{R}$ and its converse are INCLUSION simulations, equivalently when $\mathcal{R}$ and its converse are REFINEMENT simulations, equivalently when $\mathcal{R}$ and its converse are CONVEX simulations.

We illustrate these distinctions with an example taken from [Pit01]. Consider the two root nodes in Fig. 11. They are neither lower bisimilar nor upper bisimilar. But they are mutually convex similar, by the following calculation:

$$\begin{aligned}
&a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge}\\
&\quad= \quad a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge\ or\ }a.\mathsf{diverge}\\
&\quad\leqslant \quad a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge\ or\ }a.(a.\mathsf{diverge\ or\ }a.a.\mathsf{diverge})\\
&a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge}\\
&\quad= \quad a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge}\\
&\quad= \quad a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge\ or\ }a.(a.a.\mathsf{diverge\ or\ }a.a.\mathsf{diverge})\\
&\quad\geqslant \quad a.\mathsf{diverge\ or\ }a.a.a.\mathsf{diverge\ or\ }a.(a.\mathsf{diverge\ or\ }a.a.\mathsf{diverge})
\end{aligned}$$

using the fact that divergence is least in the convex ordering. This argument also demonstrates the equality of these processes in any domain-theoretic semantics, such as in [Abr91].

As a final example, in the setting of transition systems with divergence and deadlock, the paper [AH92] defines simulation corresponding to the NDBP illustrated in Fig. 12.

4.4. **Modal Logic For Similarity.** It is well known [HM80] that similarity and bisimilarity can be characterized using Hennessy-Milner logic. We want to adapt this theory in the presence of divergence and other errors. For a finite set E, the *formulas* for E-errors over a set Act of actions are defined inductively

$$\begin{array}{rcll} P & ::= & \triangledown^{\mathcal{D}} & (\mathcal{D} \subseteq \mathcal{P}E) \\ & \mid & \Diamond^{\mathcal{D}}_{\mathcal{D}'}a\ \bigwedge_{i\in I} P_i & (\mathcal{D},\mathcal{D}' \subseteq \mathcal{P}E,\ \mathcal{D}\cap\mathcal{D}' = \emptyset,\ a \in \mathsf{Act},\ I \text{ countable}) \\ & \mid & \Box^{\mathcal{D}}_{\mathcal{D}'}a\ \bigvee_{i\in I} P_i & (\mathcal{D},\mathcal{D}' \subseteq \mathcal{P}E,\ \mathcal{D}\cap\mathcal{D}' = \emptyset,\ a \in \mathsf{Act},\ I \text{ countable}) \end{array}$$

We define a relation $x \vDash P$, where x is a node of an LTS with E-errors over Act, and P is a formula for E-errors over Act, by induction on P.

- $x \vDash \triangledown^{\mathcal{D}}$ when $\mathsf{Errors}(x) \in \mathcal{D}$
- if $\mathsf{Errors}(x) \in \mathcal{D}$ then $x \vDash \Diamond^{\mathcal{D}}_{\mathcal{D}'}a \bigwedge_{i\in I} P_i$ and $x \vDash \Box^{\mathcal{D}}_{\mathcal{D}'}a \bigvee_{i\in I} P_i$
- if $\mathsf{Errors}(x) \in \mathcal{D}'$ then $x \nvDash \Diamond^{\mathcal{D}}_{\mathcal{D}'}a \bigwedge_{i\in I} P_i$ and $x \nvDash \Box^{\mathcal{D}}_{\mathcal{D}'}a \bigvee_{i\in I} P_i$
- if $\mathsf{Errors}(x) \notin \mathcal{D}\cup\mathcal{D}'$ then
  - $x \vDash \Diamond^{\mathcal{D}}_{\mathcal{D}'}a \bigwedge_{i\in I} P_i$ when there is $x \xrightarrow{a} y$ such that $y \vDash P_i$ for all $i \in I$
  - $x \vDash \Box^{\mathcal{D}}_{\mathcal{D}'}a \bigvee_{i\in I} P_i$ when for all $x \xrightarrow{a} y$ there is $i \in I$ such that $y \vDash P_i$.

The cases of $\Diamond^{\mathcal{D}}_{\mathcal{D}'}$ and $\Box^{\mathcal{D}}_{\mathcal{D}'}$ in which $\mathcal{D}' = (\mathcal{P}E)\setminus\mathcal{D}$ are redundant, because each is equivalent to $\triangledown^{\mathcal{D}}$. So the set of *modalities* for E-errors is defined as

$$\begin{array}{rcl} \mathrm{modal}(E) & \stackrel{\mathrm{def}}{=} & (\{\triangledown^{\mathcal{D}} \mid \mathcal{D} \subseteq E\} \\ & & \cup\{\Diamond^{\mathcal{D}}_{\mathcal{D}} \mid \mathcal{D},\mathcal{D}' \subseteq E, \mathcal{D}\cap\mathcal{D}' = \emptyset\} \\ & & \cup\{\Box^{\mathcal{D}}_{\mathcal{D}} \mid \mathcal{D},\mathcal{D}' \subseteq E, \mathcal{D}\cap\mathcal{D}' = \emptyset\})\ /\ \equiv \end{array}$$

where $\equiv$ identifies $\Diamond^{\mathcal{D}}_{\mathcal{P}E\setminus\mathcal{D}}$ and $\Box^{\mathcal{D}}_{\mathcal{P}E\setminus\mathcal{D}}$ with $\triangledown^{\mathcal{D}}$. We abbreviate

$$\mathrm{T} \stackrel{\mathrm{def}}{=} \triangledown^{\mathcal{P}E} \qquad \mathrm{F} \stackrel{\mathrm{def}}{=} \triangledown^{\emptyset} \tag{4.11}$$

$$\Diamond \stackrel{\mathrm{def}}{=} \Diamond^{\emptyset}_{\emptyset} \qquad \Box \stackrel{\mathrm{def}}{=} \Box^{\emptyset}_{\emptyset} \tag{4.12}$$

so the modalities for no errors are $\mathrm{T}, \mathrm{F}, \Diamond, \Box$. Where $\mathsf{d} \in E$ we abbreviate

$$\Uparrow \stackrel{\mathrm{def}}{=} \triangledown^{\{D\subseteq E \mid \mathsf{d}\in D\}} \qquad \not\Uparrow \stackrel{\mathrm{def}}{=} \triangledown^{\{D\subseteq E \mid \mathsf{d}\notin D\}} \tag{4.13}$$

$$\Diamond^{\wedge\Uparrow} \stackrel{\mathrm{def}}{=} \Diamond^{\emptyset}_{\{D\subseteq E \mid \mathsf{d}\notin D\}} \qquad \Box^{\wedge\Uparrow} \stackrel{\mathrm{def}}{=} \Box^{\emptyset}_{\{D\subseteq E \mid \mathsf{d}\notin D\}} \tag{4.14}$$

$$\Diamond^{\wedge\not\Uparrow} \stackrel{\mathrm{def}}{=} \Diamond^{\emptyset}_{\{D\subseteq E \mid \mathsf{d}\in D\}} \qquad \Box^{\wedge\not\Uparrow} \stackrel{\mathrm{def}}{=} \Box^{\emptyset}_{\{D\subseteq E \mid \mathsf{d}\in D\}} \tag{4.15}$$

$$\Diamond^{\vee\Uparrow} \stackrel{\mathrm{def}}{=} \Diamond^{\{D\subseteq E \mid \mathsf{d}\in D\}}_{\emptyset} \qquad \Box^{\vee\Uparrow} \stackrel{\mathrm{def}}{=} \Box^{\{D\subseteq E \mid \mathsf{d}\in D\}}_{\emptyset} \tag{4.16}$$

$$\Diamond^{\vee\not\Uparrow} \stackrel{\mathrm{def}}{=} \Diamond^{\{D\subseteq E \mid \mathsf{d}\notin D\}}_{\emptyset} \qquad \Box^{\vee\not\Uparrow} \stackrel{\mathrm{def}}{=} \Box^{\{D\subseteq E \mid \mathsf{d}\notin D\}}_{\emptyset} \tag{4.17}$$

so the modalities for divergence are

$$\mathrm{T}, \mathrm{F}, \Uparrow, \not\Uparrow, \Diamond, \Diamond^{\wedge\Uparrow}, \Diamond^{\wedge\not\Uparrow}, \Diamond^{\vee\Uparrow}, \Diamond^{\vee\not\Uparrow}, \Box, \Box^{\wedge\Uparrow}, \Box^{\wedge\not\Uparrow}, \Box^{\vee\Uparrow}, \Box^{\vee\not\Uparrow}$$

Definition 4.12. Let E be a finite set. Let R be a set of modalities for E-errors, i.e. $R \subseteq \mathrm{modal}(E)$. It induces a preorder $\leqslant^{R}_{\mathsf{Act}}$ on nodes of LTSs with E-errors over Act as follows: $x \leqslant^{R}_{\mathsf{Act}} y$ when, for any *R-formula* P over Act—i.e. formula using only modalities in R—if $x \vDash P$ then $y \vDash P$.

For a given NDBP $\sqsubseteq$, we want to know when a set R of modalities characterizes $\sqsubseteq$ similarity. In keeping with the theme of the paper, we shall reduce this question to boolean conditions on R and $\sqsubseteq$ that can be mechanically checked.

Definition 4.13. Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. A modality $c \in \text{modal}(E)$ is *sound* for $\sqsubseteq$ under the following conditions.

- For $c = \nabla^{\mathcal{D}}$, when $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D \in \mathcal{D}$ imply $D' \in \mathcal{D}$.
- For $c = \Diamond^{\mathcal{D}}_{\mathcal{D}}$, when all the following hold:
  - $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D \in \mathcal{D}$ imply $D' \in \mathcal{D}$
  - $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D' \in \mathcal{D}'$ imply $D \in \mathcal{D}'$
  - $\{\mathsf{t}, \mathsf{f}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ implies either $D \in \mathcal{D}'$ or $D' \in \mathcal{D}$.
- For $c = \Box^{\mathcal{D}}_{\mathcal{D}}$, when all the following hold:
  - $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D \in \mathcal{D}$ imply $D' \in \mathcal{D}$
  - $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D' \in \mathcal{D}'$ imply $D \in \mathcal{D}'$
  - $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ implies either $D \in \mathcal{D}'$ or $D' \in \mathcal{D}$.

We note that, for a finite set E of errors,

- only T and F are sound for INCONSISTENT
- all modalities are sound for EQUALITY.

The sound connectives for all the NDBPs for no errors, and for all the NDBPs for divergence, are shown in Fig. 13–14.

Proposition 4.14. *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. For $R \subseteq \text{modal}(E)$, the following are equivalent.*

1. *Each modality in R is sound for $\sqsubseteq$.*
2. *For every set Act of actions and nodes x, x' of countably branching LTSs with E-errors over Act, if $x \lesssim^{\sqsubseteq}_{\mathsf{Act}} x'$ then $x \leqslant^{R}_{\mathsf{Act}} x'$.*

Proof. **(1) ⇒ (2):** Suppose $x \lesssim^{\sqsubseteq}_{\mathsf{Act}} x'$. We have to show that $x \vDash P$ implies $x' \vDash P$, where P is an R-formula over Act. We proceed by induction on P.

- Suppose $P = \nabla^{\mathcal{D}}$, so $\mathsf{Errors}(x) \in \mathcal{D}$. Then $\{\mathsf{t}\} + \mathsf{Errors}(x) \sqsubseteq \{\mathsf{t}\} + \mathsf{Errors}(x')$. So the soundness of $\nabla^{\mathcal{D}}$ gives $\mathsf{Errors}(x') \in \mathcal{D}$.
- Suppose $P = \Diamond^{\mathcal{D}}_{\mathcal{D}} \bigwedge_{i \in I} P_i$.
  - If $x \in \mathsf{Errors}(\mathcal{D})$, then $\{\mathsf{t}\} + \mathsf{Errors}(x) \sqsubseteq \{\mathsf{t}\} + \mathsf{Errors}(x')$. So the soundness of $\Diamond^{\mathcal{D}}_{\mathcal{D}'}$ gives $\mathsf{Errors}(x') \in \mathcal{D}$.
  - Otherwise $\mathsf{Errors}(x) \notin \mathcal{D}'$, so $\{\mathsf{t}\} + \mathsf{Errors}(x) \sqsubseteq \{\mathsf{t}\} + \mathsf{Errors}(x')$ and the soundness of $\Diamond^{\mathcal{D}}_{\mathcal{D}'}$ give $\mathsf{Errors}(x') \notin \mathcal{D}'$, and $x \xrightarrow{a} y$ for some y such that P_i for all $i \in I$. There are two cases.
    - If $\{\mathsf{t}, \mathsf{f}\} + \mathsf{Errors}(x) \sqsubseteq \{\mathsf{t}\} + \mathsf{Errors}(x')$ then soundness of $\Diamond^{\mathcal{D}}_{\mathcal{D}'}$ gives $\mathsf{Errors}(x') \in \mathcal{D}$.
    - Otherwise $x' \xrightarrow{a} y'$ for some y' such that $y \lesssim^{\sqsubseteq}_{R} y'$. For each $i \in I$, the inductive hypothesis at P_i gives $y' \vDash P_i$.
- The case $P = \Box^{\mathcal{D}}_{\mathcal{D}} \bigwedge_{i \in I} P_i$ is treated dually.

(2) ⇒ (1): Let $c \in R$ be unsound. We set $\mathsf{Act} = \{a, b\}$ with $a \neq b$, and show there exist nodes x, x' of countably branching LTSs with E-errors over Act, and an R-formula P over Act, such that the relation $\{(x, x')\}$ is a $\sqsubseteq$-simulation and $x \vDash P$ but $x' \nvDash P$. Hence $x \lesssim^{\sqsubseteq}_{\mathsf{Act}} x'$ but $x \not\leqslant^{R}_{\mathsf{Act}} x'$.

NDBP	sound modalities	a set of sound modalities is complete when it contains
INCONSISTENT	T, F	(no requirement)
EQUALITY	$\mathrm{T}, \mathrm{F}, \Diamond, \Box$	$\Diamond$ and $\Box$
INCLUSION	$\mathrm{T}, \mathrm{F}, \Diamond$	$\Diamond$
REFINEMENT	$\mathrm{T}, \mathrm{F}, \Box$	$\Box$

Figure 13: Sound and complete modal logics for similarity, wrt NDBPs for no errors

- Suppose $c = \nabla^{\mathcal{D}}$, so there is D, D' such that $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D \in \mathcal{D}$ but $D' \notin \mathcal{D}$. Let x be a node with $\mathsf{Errors}(x) = D$ and sole transition $x \xrightarrow{a} x$, let x' be a node with $\mathsf{Errors}(x') = D'$ and sole transition $x \xrightarrow{a} x'$, and let $P \stackrel{\text{def}}{=} \nabla^{\mathcal{D}}$.
- Suppose $c = \Diamond^{\mathcal{D}}_{\mathcal{D}'}$. Then there are three possibilities.
  - Suppose there is D, D' such that $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D \in \mathcal{D}, D' \notin \mathcal{D}$. Let x be a node with $\mathsf{Errors}(x) = D$ and sole transition $x \xrightarrow{a} x$, let x' be a node with h $\mathsf{Errors}(x') = D'$ and sole transition $x \xrightarrow{a} x'$, and let $P \stackrel{\text{def}}{=} \Diamond^{\mathcal{D}}_{\mathcal{D}'} b \bigwedge_{\emptyset}$.
  - Suppose there is D, D' such that $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D \notin \mathcal{D}', D' \in \mathcal{D}'$. Then let x be a node with $\mathsf{Errors}(x) = D$ and sole transition $x \xrightarrow{a} x$, let x' be a node with h $\mathsf{Errors}(x') = D'$ and sole transition $x \xrightarrow{a} x'$, and let $P \stackrel{\text{def}}{=} \Diamond^{\mathcal{D}}_{\mathcal{D}'} a \bigwedge_{\emptyset}$.
  - Suppose there is D, D' such that $\{\mathsf{t}, \mathsf{f}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ and $D \notin \mathcal{D}', D' \notin \mathcal{D}$. Then let x be a node with $\mathsf{Errors}(x) = D$ and sole transitions $x \xrightarrow{a} x$ and $x \xrightarrow{b} x$, and let x' be a node with h $\mathsf{Errors}(x') = D'$ and sole transition $x \xrightarrow{a} x'$, and let $P \stackrel{\text{def}}{=} \Diamond^{\mathcal{D}}_{\mathcal{D}'} b \bigwedge_{\emptyset}$.
- The case $c = \Box^{\mathcal{D}}_{\mathcal{D}'}$ is treated dually.

Definition 4.15. Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. Let R be a set of modalities that are sound for $\sqsubseteq$. Then R is *complete* for $\sqsubseteq$ when all the following hold:

(1) if $\{\mathsf{t}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$ then there is $\nabla^{\mathcal{D}} \in R$ such that $D \in \mathcal{D}, D' \notin \mathcal{D}$
(2) if $\{\mathsf{t}, \mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$ then there is $\Diamond^{\mathcal{D}}_{\mathcal{D}'} \in R$ such that $D \notin \mathcal{D}', D' \notin \mathcal{D}$
(3) if $\{\mathsf{t}\} + D \not\sqsubseteq \{\mathsf{t}, \mathsf{f}\} + D'$ then there is $\Box^{\mathcal{D}}_{\mathcal{D}'} \in R$ such that $D \notin \mathcal{D}', D' \notin \mathcal{D}$.

The complete sets of connectives for all the NDBPs for no errors, and all the NDBPs for divergence, are shown in Fig. 13–14.

Proposition 4.16. *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. Let R be a set of modalities that are sound for $\sqsubseteq$. Then the following are equivalent.*

(1) *R is complete for $\sqsubseteq$.*
(2) *For any set Act of actions and nodes x, x' of countably branching LTSs with E-errors over Act, if $x \lesssim^{R}_{\mathsf{Act}} x'$ then $x \lesssim^{\sqsubseteq}_{\mathsf{Act}} x'$.*

Proof.

(1) ⇒ (2): We show that $\leqslant^{R}_{\mathsf{Act}}$ is a $\sqsubseteq$ simulation. Suppose $x \leqslant^{R}_{\mathsf{Act}} x'$.

NDBP	sound modalities	a set of sound modalities is complete when it contains
INCONSISTENT	T,F	(no requirement)
EQUALITY	$\mathrm{T},\mathrm{F},\Uparrow,\not\Uparrow,$ $\Diamond,\Diamond^{\wedge\Uparrow},\Diamond^{\wedge\not\Uparrow},\Diamond^{\vee\Uparrow},\Diamond^{\vee\not\Uparrow},$ $\Box,\Box^{\wedge\Uparrow},\Box^{\wedge\not\Uparrow},\Box^{\vee\Uparrow},\Box^{\vee\not\Uparrow}$	$\Uparrow$ and $\not\Uparrow$ and ($\Diamond$ or (($\Diamond^{\wedge\Uparrow}$ or $\Diamond^{\vee\not\Uparrow}$) and ($\Diamond^{\wedge\not\Uparrow}$ or $\Diamond^{\vee\Uparrow}$))) and ($\Box$ or (($\Box^{\wedge\Uparrow}$ or $\Box^{\vee\not\Uparrow}$) and ($\Box^{\wedge\not\Uparrow}$ or $\Box^{\vee\Uparrow}$)))
LOWER	$\mathrm{T},\mathrm{F},\Diamond$	$\Diamond$
OP-LOWER	$\mathrm{T},\mathrm{F},\Box$	$\Box$
UPPER	$\mathrm{T},\mathrm{F},\not\Uparrow,\Box^{\wedge\not\Uparrow}$	$\not\Uparrow$ and $\Box^{\wedge\not\Uparrow}$
OP-UPPER	$\mathrm{T},\mathrm{F},\Uparrow,\Diamond^{\vee\Uparrow}$	$\Uparrow$ and $\Diamond^{\vee\Uparrow}$
SMASH	$\mathrm{T},\mathrm{F},\not\Uparrow,\Diamond^{\wedge\not\Uparrow},\Box^{\wedge\not\Uparrow}$	$\not\Uparrow$ and $\Diamond^{\wedge\not\Uparrow}$ and $\Box^{\wedge\not\Uparrow}$
OP-SMASH	$\mathrm{T},\mathrm{F},\Uparrow,\Diamond^{\vee\Uparrow},\Box^{\vee\Uparrow}$	$\Uparrow$ and $\Diamond^{\vee\Uparrow}$ and $\Box^{\vee\Uparrow}$
CONVEX	$\mathrm{T},\mathrm{F},\not\Uparrow,\Box^{\wedge\not\Uparrow},$ $\Diamond,\Diamond^{\wedge\not\Uparrow},\Diamond^{\vee\not\Uparrow}$	$\not\Uparrow$ and $\Box^{\wedge\not\Uparrow}$ and ($\Diamond$ or $\Diamond^{\wedge\not\Uparrow}$)
OP-CONVEX	$\mathrm{T},\mathrm{F},\Uparrow,\Diamond^{\vee\Uparrow},$ $\Box,\Box^{\wedge\Uparrow},\Box^{\vee\Uparrow}$	$\Uparrow$ and $\Diamond^{\vee\Uparrow}$ and ($\Box$ or $\Box^{\vee\Uparrow}$)
INCLUSION	$\mathrm{T},\mathrm{F},\Uparrow,\Diamond,\Diamond^{\wedge\Uparrow},\Diamond^{\vee\Uparrow}$	$\Uparrow$ and $\Diamond$
REFINEMENT	$\mathrm{T},\mathrm{F},\not\Uparrow,\Box,\Box^{\vee\not\Uparrow},\Box^{\wedge\not\Uparrow}$	$\not\Uparrow$ and $\Box$
SESQUI INCLUSION	$\mathrm{T},\mathrm{F},\Uparrow,\Diamond,\Diamond^{\wedge\Uparrow},\Diamond^{\vee\Uparrow},$ $\Box,\Box^{\wedge\Uparrow},\Box^{\vee\Uparrow}$	$\Uparrow$ and $\Diamond$ and $\Box$
SESQUI REFINEMENT	$\mathrm{T},\mathrm{F},\not\Uparrow,\Diamond,\Diamond^{\wedge\not\Uparrow},\Diamond^{\vee\not\Uparrow},$ $\Box,\Box^{\wedge\not\Uparrow},\Box^{\vee\not\Uparrow}$	$\not\Uparrow$ and $\Diamond$ and $\Box$
PLUCKED	$\mathrm{T},\mathrm{F},\Uparrow,\not\Uparrow,$ $\Diamond,\Diamond^{\wedge\Uparrow},\Diamond^{\wedge\not\Uparrow},\Diamond^{\vee\Uparrow},\Diamond^{\vee\not\Uparrow},$ $\Box^{\wedge\not\Uparrow},\Box^{\vee\Uparrow}$	$\Uparrow$ and $\not\Uparrow$ and ($\Diamond$ or (($\Diamond^{\wedge\Uparrow}$ or $\Diamond^{\vee\not\Uparrow}$) and ($\Diamond^{\wedge\not\Uparrow}$ or $\Diamond^{\vee\Uparrow}$))) and ($\Box^{\wedge\not\Uparrow}$ or $\Box^{\vee\Uparrow}$)
OP-PLUCKED	$\mathrm{T},\mathrm{F},\Uparrow,\not\Uparrow,$ $\Diamond^{\wedge\not\Uparrow},\Diamond^{\vee\Uparrow},$ $\Box,\Box^{\wedge\Uparrow},\Box^{\wedge\not\Uparrow},\Box^{\vee\Uparrow},\Box^{\vee\not\Uparrow}$	$\Uparrow$ and $\not\Uparrow$ and ($\Diamond^{\wedge\not\Uparrow}$ or $\Diamond^{\vee\Uparrow}$) and ($\Box$ or (($\Box^{\wedge\Uparrow}$ or $\Box^{\vee\not\Uparrow}$) and ($\Box^{\wedge\not\Uparrow}$ or $\Box^{\vee\Uparrow}$)))
STUNTED	$\mathrm{T},\mathrm{F},\Uparrow,\Diamond,\Diamond^{\wedge\Uparrow},\Diamond^{\vee\Uparrow},\Box^{\vee\Uparrow}$	$\Uparrow$ and $\Diamond$ and $\Box^{\vee\Uparrow}$
OP-STUNTED	$\mathrm{T},\mathrm{F},\not\Uparrow,\Diamond^{\wedge\not\Uparrow},\Box,\Box^{\wedge\not\Uparrow},\Box^{\vee\not\Uparrow}$	$\not\Uparrow$ and $\Diamond^{\wedge\not\Uparrow}$ and $\Box$
LOWER CONGRUENCE	$\mathrm{T},\mathrm{F},\Diamond,\Box$	$\Diamond$ and $\Box$
UPPER CONGRUENCE	$\mathrm{T},\mathrm{F},\Uparrow,\not\Uparrow,$ $\Diamond^{\wedge\not\Uparrow},\Diamond^{\vee\Uparrow},$ $\Box^{\wedge\not\Uparrow},\Box^{\vee\Uparrow}$	$\Uparrow$ and $\not\Uparrow$ and ($\Diamond^{\wedge\not\Uparrow}$ or $\Diamond^{\vee\Uparrow}$) and ($\Box^{\wedge\not\Uparrow}$ or $\Box^{\vee\Uparrow}$)

Figure 14: Sound and complete modal logics for similarity, wrt NDBPs for divergence

- If $\{\mathsf{t}\}+\mathsf{Errors}(x)\not\sqsubseteq\{\mathsf{t}\}+\mathsf{Errors}(x')$, then there is $\triangledown^{\mathcal{D}}\in R$ such that $\mathsf{Errors}(x)\in\mathcal{D},\mathsf{Errors}(x)\notin\mathcal{D}'$. Let P be the R-formula $\triangledown^{\mathcal{D}}$, giving $x\models P$ but $x'\not\models P$.
- If $\{\mathsf{t},\mathsf{f}\}+\mathsf{Errors}(x)\not\sqsubseteq\{\mathsf{t}\}+\mathsf{Errors}(x')$ then there is $\Diamond^{\mathcal{D}}_{\mathcal{D}'}\in R$ such that $\mathsf{Errors}(x)\notin\mathcal{D}',\mathsf{Errors}(x')\notin\mathcal{D}$. Suppose that $x\xrightarrow{a}y$. For each y' such that $x'\xrightarrow{a}y'$, suppose for a contradiction that $y\not\leq^{R}_{\mathsf{Act}}y'$, so we pick an R-formula $P_{y'}$ such

that $y \vDash P_{y'}$ but $y' \nvDash P_{y'}$. Let P be the R-formula

$$\Diamond^{\mathcal{D}}_{\mathcal{D}'} \bigwedge_{x' \xrightarrow{a} y'} P_{y'}$$

giving $x \vDash P$ but $x' \nvDash P$.

- The case $\{\mathsf{t}\} + \mathsf{Errors}(x) \not\sqsubseteq \{\mathsf{t},\mathsf{f}\} + \mathsf{Errors}(x')$ is treated dually.

(2) $\Rightarrow$ (1): Suppose R is incomplete, i.e. one of conditions (1)–(3) within Def. 4.15 fails. In each case, we give a set Act of actions and nodes x, x' of countably branching LTSs with E-errors over Act such that $x \leqslant^{R}_{\mathsf{Act}} x'$ but $x \not\lesssim^{\sqsubseteq}_{\mathsf{Act}} x'$.

- Suppose (1) fails. Then there is D, D' such that $\{\mathsf{t}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$ but there does not exist $\triangledown^{\mathcal{D}} \in R$ such that $D \in \mathcal{D}, D' \notin \mathcal{D}$. Let $\mathsf{Act} \stackrel{\text{def}}{=} \emptyset$, let x be a node with $\mathsf{Errors}(x) = D$ and no transitions, and let x' be a node with $\mathsf{Errors}(x) = D'$ and no transitions. Then $x \not\lesssim^{\sqsubseteq}_{\mathsf{Act}} x'$. Let P be an R-formula over Act such that $x \vDash P$. Since there are no actions, P must be of the form $\triangledown^{\mathcal{D}}$, so $D \in \mathcal{D}$. By assumption $D' \in \mathcal{D}$, so $y \vDash P$.
- Suppose (1) holds but (2) fails. Then there is D, D' such that $\{\mathsf{t},\mathsf{f}\} + D \not\sqsubseteq \{\mathsf{t}\} + D'$ but there does not exist $\Diamond^{\mathcal{D}}_{\mathcal{D}'} \in R$ such that $D \notin \mathcal{D}', D' \notin \mathcal{D}$. We know $\{\mathsf{t}\} + D \sqsubseteq \{\mathsf{t}\} + D'$ because otherwise (1) gives $\triangledown^{\mathcal{D}} = \Diamond^{\mathcal{D}}_{(\mathcal{P}E)\setminus\mathcal{D}} \in R$ with $D \in \mathcal{D}, D' \notin \mathcal{D}$. We set $\mathsf{Act} = \{a\}$. Let x be a node with $\mathsf{Errors}(x) = D$ and sole transition $x \xrightarrow{a} x$ and let x' be a node with $\mathsf{Errors}(x') = D'$ and no transitions. Let P be an R-formula over Act such that $x \vDash P$.
  - If $P = \triangledown^{\mathcal{D}}$ then $D \in \mathcal{D}$. Since $\triangledown^{\mathcal{D}} \in R$, it is sound for $\sqsubseteq$, so $D' \in \mathcal{D}$ i.e. $x' \vDash P$.
  - If $P = \Diamond^{\mathcal{D}}_{\mathcal{D}'} a \bigwedge_{i \in I} P_i$ then $D \notin \mathcal{D}'$. Since $\Diamond^{\mathcal{D}}_{\mathcal{D}'} \in R$, our assumption gives $D' \in \mathcal{D}$ so $x' \vDash P$.
  - If $P = \Box^{\mathcal{D}}_{\mathcal{D}'} a \bigvee_{i \in I} P_i$ then $D \notin \mathcal{D}'$. Since $\Box^{\mathcal{D}}_{\mathcal{D}'} \in R$, it is sound for $\sqsubseteq$, so $D' \notin \mathcal{D}'$, so $x' \vDash P$.
- The case where (1) holds but (3) fails is treated dually.

Example 4.17. In [Abr91] a modal logic is given to characterize convex similarity (there called "partial bisimilarity") over nonempty Act. The logic (cf. [Mil81, Sti87]) provides two modalities, viz. $\Diamond$ and $\Box^{\wedge\not\uparrow}$. If Act were allowed to be empty, then the modality $\not\uparrow$ would also be required, in accordance with Fig. 14.

Example 4.18. In [AH92] a modal logic is given to characterize ACETO-HENNESSY similarity (the NDBP shown in Fig. 11). The logic provides four modalities, viz.

$$\triangledown^{\{\{\}\}}, \triangledown^{\{\{\mathsf{c}\}\}}, \Diamond, \Box^{\wedge\not\uparrow}$$

It is easily verified that this is a complete set of sound modalities for ACETO-HENNESSY.

One might wonder whether an NDBP has to have a complete set of sound modalities. In fact, there are two canonical such sets, called the *positive modalities* and the *negative modalities*.

Proposition 4.19. *Let E be a finite set, and let $\sqsubseteq$ be a NDBP for E-errors. Then each of the following is a complete set of sound modalities for $\sqsubseteq$.*

$$\begin{aligned}
\mathrm{modal}^{+}(\sqsubseteq) \;&\stackrel{\mathrm{def}}{=}\; \{\triangledown^{\{F\subseteq E \,|\, \{\mathsf{t}\}+D \sqsubseteq \{\mathsf{t}\}+F\}} \mid D\subseteq E\} \\
&\quad \cup \{\Diamond^{\{F\subseteq E \,|\, \{\mathsf{t},\mathsf{f}\}+D \sqsubseteq \{\mathsf{t}\}+F\}}_{\{F\subseteq E \,|\, \{\mathsf{t}\}+D \not\sqsubseteq \{\mathsf{t}\}+F\}} \mid D\subseteq E\} \\
&\quad \cup \{\Box^{\{F\subseteq E \,|\, \{\mathsf{t}\}+D \sqsubseteq \{\mathsf{t},\mathsf{f}\}+F\}}_{\{F\subseteq E \,|\, \{\mathsf{t}\}+D \not\sqsubseteq \{\mathsf{t}\}+F\}} \mid D\subseteq E\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{modal}^{-}(\sqsubseteq) \;&\stackrel{\mathrm{def}}{=}\; \{\triangledown^{\{F\subseteq E \,|\, \{\mathsf{t}\}+F \not\sqsubseteq \{\mathsf{t}\}+D\}} \mid D\subseteq E\} \\
&\quad \cup \{\Diamond^{\{F\subseteq E \,|\, \{\mathsf{t}\}+F \not\sqsubseteq \{\mathsf{t}\}+D\}}_{\{F\subseteq E \,|\, \{\mathsf{t},\mathsf{f}\}+F \sqsubseteq \{\mathsf{t}\}+D\}} \mid D\subseteq E\} \\
&\quad \cup \{\Box^{\{F\subseteq E \,|\, \{\mathsf{t}\}+F \not\sqsubseteq \{\mathsf{t}\}+D\}}_{\{F\subseteq E \,|\, \{\mathsf{t}\}+F \sqsubseteq \{\mathsf{t},\mathsf{f}\}+D\}} \mid D\subseteq E\}
\end{aligned}$$

Hence the set of all sound modalities for $\sqsubseteq$ is complete. □

5. Conclusions

We have seen that the notion of boolean precongruence is extremely helpful as a parameter of our analysis, whether in the deterministic or nondeterministic setting. It determines

- the ordering at each ground type
- whether a semilattice is compatible with these orderings
- whether amb is monotone
- a contextual preorder
- whether contexts of Sierpinski type suffice
- whether contexts of zero type suffice
- a way of lifting a relation
- a power-poset construction
- an appropriate definition of simulation
- a modal logic (indeed, a collection of modal logics) characterizing similarity.

Each of these things is familiar from the literature, but our use of a boolean precongruence provides a more systematic analysis.

Although our primary case of interest is where the only error is divergence, we have seen that the entire theory works more generally, for any finite set E of errors. This helps to elucidate the theory, and moreover has applications to languages involving behaviours such as crash or deadlock, both in deterministic settings [CCF94, Lai07a] and nondeterministic ones [Lai06, Ros98, AH92].

We have not considered domain theory in this paper; a task that remains is to develop the *smash powerdomain* of a pointed dcpo. It differs from the convex powerdomain in that all elements containing $\perp$ must be identified with $\perp$.

Acknowledgements

I am grateful to Soren Lassen for initiating the research that led to this paper, and extensive discussion. I thank Jim Laird for confirming the claims about contextual preorders in Sect. 3.1.

References

[Abr91] S Abramsky. A domain equation for bisimulation. *Information and Computation*, 92(2), 1991.

[AH92] Luca Aceto and Matthew Hennessy. Termination, deadlock, and divergence. *Journal of the ACM*, 39(1):147–187, 1992.

[AP97] S. O. Anderson and John Power. A representable approach to finite nondeterminism. *Theor. Comput. Sci*, 177(1):3–25, 1997.

[Bal00] Alexandru Baltag. A logic for coalgebraic simulation. *Electr. Notes Theor. Comput. Sci*, 33, 2000.

[BL95] Gérard Boudol and Cosimo Laneve. Termination, deadlock and divergence in the lambda-calculus with multiplicities. *Electr. Notes Theor. Comput. Sci*, 1, 1995.

[CCF94] R. Cartwright, P.-L. Curien, and M. Felleisen. Fully abstract semantics for observably sequential languages. *Information and Computation*, 111(2):297–401, June 1994.

[HJ04] Jesse Hughes and Bart Jacobs. Simulations in coalgebra. *Theor. Comput. Sci*, 327(1-2):71–108, 2004.

[HM80] Matthew Hennessy and Robin Milner. On observing nondeterminism and concurrency. In *ICALP*, pages 299–309, 1980.

[How89] Douglas J. Howe. Equality in lazy computation systems. In *Proc. 4th IEEE Symposium on Logic in Computer Science*, pages 198–203, 1989.

[How96] D J Howe. Proving congruence of bisimulation in functional programming languages. *Inf. and Comp.*, 124(2), 1996.

[HP80] M. Hennessy and G. D. Plotkin. A term model for CCS. In P. Dembiński, editor, 9^{th} *Symposium on Mathematical Foundations of Computer Science*, volume 88 of *Lecture Notes in Computer Science*, pages 261–274. Springer-Verlag, 1980.

[HT00] Wim H. Hesselink and Albert Thijs. Fixpoint semantics and simulation. *Theor. Comput. Sci*, 238(1-2):275–311, 2000.

[Lai05] James Laird. Locally boolean domains. *Theor. Comput. Sci*, 342(1):132–148, 2005.

[Lai06] J D Laird. Bidomains and full abstraction for countable nondeterminism. In L. Aceto and A. Ingólfsdóttir, editors, *FoSSaCS*, volume 3921 of *LNCS*. Springer, 2006.

[Lai07a] James Laird. Bistable biorders: A sequential domain theory. *Logical Methods in Computer Science*, 3(2), 2007.

[Lai07b] Jim Laird. Full abstraction for recursive types with control and countable nondeterminism. Draft, 2007.

[Lai09] James Laird. Nondeterminism and observable sequentiality. In Erich Grädel and Reinhard Kahle, editors, *CSL*, volume 5771 of *Lecture Notes in Computer Science*, pages 379–393. Springer, 2009.

[Las98] S B Lassen. *Relational Reasoning about Functions and Nondeterminism*. PhD thesis, Univ. of Aarhus, 1998.

[LL09] S. B. Lassen and P. B. Levy. Labelled transition systems over a game graph. Contributed talk, Games for Logic and Programming IV, York, England, 2009.

[LLP05] S B Lassen, P B Levy, and P Panangaden. Divergence-least semantics of `amb` is Hoare. 3rd APPSEM II workshop, Frauenchiemsee, Germany, 2005.

[Mil81] Robin Milner. A modal characterization of observable machine-behaviour. In E. Astesiano and C. Böhm, editors, *Proceedings CAAP '81*, volume 112 of *LNCS*, pages 25–34, Genoa, March 1981. Springer-Verlag.

[Mor98] A. K. Moran. *Call-by-name, Call-by-need, and McCarthy's Amb*. PhD thesis, Department of Computing Science, Chalmers University of Technology and University of Gothenburg, Gothenburg, Sweden, September 1998.

[NH84] Rocco De Nicola and Matthew Hennessy. Testing equivalences for processes. *Theor. Comput. Sci*, 34:83–133, 1984.

[Ong93] C. H. L. Ong. Non-determinism in a functional setting. In *Proceedings, Eighth Annual IEEE Symposium on Logic in Computer Science*, pages 275–286. IEEE Computer Society Press, 1993.

[Pit01] C Pitcher. *Functional Programming and Erratic Non-Determinism*. PhD thesis, Oxford Univ., 2001.

[Plo76] G. D. Plotkin. A powerdomain construction. *SIAM Journal of Computing*, 5(3), 1976.

[Plo77] G. D. Plotkin. LCF considered as a programming language. *Theoretical Computer Science*, 5:223–255, 1977.

[Ros92] A. W. Roscoe. An alternative order for the failures model. *Journal of Logic and Computation*, 2(5):557–577, October 1992.

[Ros93] A. W. Roscoe. Unbounded non-determinism in CSP. *Journal of Logic and Comp.*, 3(2), 1993.

[Ros98] A W Roscoe. *Theory and Practice of Concurrency*. Prentice-Hall, 1998.

[Smy78] M. B. Smyth. Power domains. *Journal of Computer and System Sciences*, 16(1):23–36, February 1978.

[Sti87] Colin Stirling. Modal logics for communicating systems. *Theoretical Computer Science*, 49(2–3):311–347, July 1987.

[Uli92] I. Ulidowski. Equivalences on observable processes. In Andre Scedrov, editor, *Proceedings of the 7th Annual IEEE Symposium on Logic in Computer Science*, pages 148–161, Santa Cruz, CA, June 1992. IEEE Computer Society Press.

[vG93] R. J. van Glabbeek. The linear time – branching time spectrum II (the semantics of sequential systems with silent moves). In E. Best, editor, *Proceedings CONCUR 93,* Hildesheim, Germany, volume 715 of *Lecture Notes in Computer Science*, pages 66–81. Springer-Verlag, 1993.

[vG01] R. J. van Glabbeek. The linear time — branching time spectrum I. The semantics of concrete, sequential processes. In J. A. Bergstra, A. Ponse, and S. A. Smolka, editors, *Handbook of Process Algebra*, pages 3–99. North-Holland, 2001.

[Wal90] D. J. Walker. Bisimulation and divergence. *Information and Computation*, 85(2):202–241, April 1990.