A tutorial on call-by-push-value

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Outline

Typed λ-calculus

We consider typed λ-calculus with boolean, function and sum types.

Types

$$A \quad ::= \quad \texttt{bool} \mid A + A \mid A \to A$$

Typing judgement $\Gamma \vdash M : A$

Terms

$$
\begin{aligned}
M \quad ::= \quad & \texttt{x} \mid \texttt{let } M \texttt{ be x.} M \\
& \mid \texttt{true} \mid \texttt{false} \mid \texttt{case } M \texttt{ of } \{\texttt{true.} M, \texttt{ false.} M\} \\
& \mid \texttt{inl } M \mid \texttt{inr } M \mid \texttt{case } M \texttt{ of } \{\texttt{inl x.} M, \texttt{ inr x.} M\} \\
& \mid \lambda \texttt{x.} M \mid MM
\end{aligned}
$$

Equational Laws

We consider the equational theory generated by the $\beta\eta$-laws.

η-law for $A \to B$

Any term $\Gamma \vdash M : A \to B$ can be expanded as

$$\lambda \texttt{x}.M\texttt{x}$$

Anything of function type is a λ-abstraction.

η-law for `bool`

Any term $\Gamma, \texttt{z} : \texttt{bool} \vdash M : B$ can be expanded as

$$\texttt{case z of } \{\texttt{true. } M[\texttt{true/z}], \texttt{ false. } M[\texttt{false/z}]\}$$

Anything of boolean type is a boolean.

The η-law for sum types is similar.

Denotational semantics in Set

A type denotes a set.

$$
\begin{aligned}
\llbracket \mathsf{bool} \rrbracket &= \mathbb{B} \overset{\mathrm{def}}{=} \{\mathsf{true}, \mathsf{false}\} \\
\llbracket A + B \rrbracket &= \llbracket A \rrbracket + \llbracket B \rrbracket \\
\llbracket A \to B \rrbracket &= \llbracket A \rrbracket \to \llbracket B \rrbracket
\end{aligned}
$$

A term $\Gamma \vdash M : B$ denotes a function $\llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} \llbracket B \rrbracket$.

Substitution Lemma

Given terms $\Gamma, \mathtt{x} : A \vdash M : B$ and $\Gamma \vdash N : A$

we can obtain $\llbracket M[N/\mathtt{x}] \rrbracket$ from $\llbracket M \rrbracket$ and $\llbracket N \rrbracket$. It is

$$
\rho \longmapsto \llbracket M \rrbracket (\rho, \mathtt{x} \mapsto \llbracket N \rrbracket \rho)
$$

Corollary

The denotational semantics validates the β and η laws.

Call-by-name evaluation of a closed term

In CBN the terminals are $\mathtt{true}, \mathtt{false}, \mathtt{inl}\ M, \mathtt{inr}\ M, \lambda\mathtt{x}.M$
To evaluate

- $\mathtt{true}$: return $\mathtt{true}$.
- $\lambda\mathtt{x}.M$: return $\lambda\mathtt{x}.M$.
- $\mathtt{inl}\ M$: return $\mathtt{inl}\ M$.
- $\mathtt{let}\ M\ \mathtt{be}\ \mathtt{x}.\ N$: evaluate $N[M/\mathtt{x}]$.
- $\mathtt{case}\ M\ \mathtt{of}\ \{\mathtt{true}.\ N, \mathtt{false}.\ N'\}$: evaluate M. If it returns $\mathtt{true}$, evaluate N, but if it returns $\mathtt{false}$, evaluate N'.
- $\mathtt{case}\ M\ \mathtt{of}\ \{\mathtt{inl}\ \mathtt{x}.\ N, \mathtt{inr}\ \mathtt{x}.\ N'\}$: evaluate M. If it returns $\mathtt{inl}\ P$, evaluate $N[P/\mathtt{x}]$, but if it returns $\mathtt{inr}\ P$, evaluate $N'[P/\mathtt{x}]$.
- MN: evaluate M. If it returns $\lambda\mathtt{x}.P$, evaluate $P[N/\mathtt{x}]$.

Call-by-value evaluation of a closed term

CBV terminals $T ::= $ `true` $|$ `false` $|$ `inl` T $|$ `inr` T $|$ $\lambda x.M$

To evaluate

- `true`: return `true`.
- $\lambda x.M$: return $\lambda x.M$.
- `inl` M: evaluate M. If it returns T, return `inl` T.
- `let` M `be x.` N: evaluate M. If it returns T, evaluate $N[T/x]$.
- `case` M `of` $\{$`true.` N`, false.` $N'\}$: evaluate M. If it returns `true`, evaluate N, but if it returns `false`, evaluate N'.
- `case` M `of` $\{$`inl x.` N`, inr x.` $N'\}$: evaluate M. If it returns `inl` T, evaluate $N[T/x]$, but if it returns `inr` T, evaluate $N'[T/x]$.
- MN: evaluate M. If it returns $\lambda x.P$, evaluate N. If that returns T, evaluate $P[T/x]$.

Adding computational effects

Errors

Let $E = \{\mathrm{CRASH}, \mathrm{BANG}, \mathrm{WALLOP}\}$ be a set of "errors". We add

$$\frac{}{\Gamma \vdash \mathtt{error}\ e : B}\ e \in E$$

To evaluate $\mathtt{error}\ e$: halt with error message e.

Printing

Let $\mathcal{A} = \{\mathrm{a, b, c, d, e}\}$ be a set of "characters". We add

$$\frac{\Gamma \vdash M : B}{\Gamma \vdash \mathtt{print}\ c.\ M : B}\ c \in \mathcal{A}$$

To evaluate $\mathtt{print}\ c.\ M$: print c and then evaluate M.

Exercises

1. Evaluate

$$\texttt{let (error CRASH) be x. } 5$$

   in CBV and CBN

2. Evaluate

$$(\lambda\texttt{x}.(x+x))(\texttt{print "hello". } 4)$$

   in CBV and CBN.

3. Evaluate

$$\texttt{case (print "hello". inr error CRASH) of}$$
$$\{\texttt{inl x. } x+1, \texttt{ inr y. } 5\}$$

   in CBV and CBN.

Big-Step Operational Semantics

We convert our CBV and CBN interpreters into big-step semantics, defined inductively.

no effects We define a relation $M \Downarrow T$ meaning M evaluates to T.

errors We define a relation $M \Downarrow T$ meaning M evaluates to T, and a relation $M \nupdownarrow e$ meaning M raises error e.

printing We define a relation $M \Downarrow m, T$ meaning M prints $m \in \mathcal{A}^*$ and finally evaluates to T.

For example, in the case of printing we have rules such as

$$\frac{}{\texttt{true} \Downarrow \varepsilon, \texttt{true}} \qquad \frac{M \Downarrow m, \texttt{true} \qquad N \Downarrow m', T}{\texttt{case } M \texttt{ of } \{\texttt{true}. N, \texttt{ false}. N'\} \Downarrow mm', T}$$

These are proved deterministic and total using Tait's method.

Observational equivalence

Two terms $\Gamma \vdash M, M' : B$ are observationally equivalent

when $\mathcal{C}[M]$ and $\mathcal{C}[M']$ have the same behaviour

for every ground (i.e. boolean) context $\mathcal{C}[\cdot]$.

Same behaviour means: print the same string, raise the same error, return the same boolean.

We write $M \simeq_{\mathbf{CBV}} M'$ and $M \simeq_{\mathbf{CBN}} M'$.

The η-law for boolean type: has it survived?

> ### η-law for `bool`
>
> Any term $\Gamma, \mathtt{z} : \mathtt{bool} \vdash M : B$ can be expanded as
>
> $$\mathtt{case\ z\ of}\ \{\mathtt{true}.\ M[\mathtt{true}/\mathtt{z}],\ \mathtt{false}.\ M[\mathtt{false}/\mathtt{z}]\}$$
>
> Anything of boolean type is a boolean.

This holds in CBV, because `z` can only be replaced by `true` or `false`.

But it's broken in CBN, because `z` might raise an error. For example,

$$\mathtt{true} \not\simeq_{\mathbf{CBN}} \mathtt{case\ z\ of}\ \{\mathtt{true}.\ \mathtt{true},\ \mathtt{false}.\ \mathtt{true}\}$$

because we can apply the context

$$\mathtt{let\ error\ CRASH\ be\ z.}\ [\cdot]$$

Similarly the η-law for sum types is valid in CBV but not in CBN.

The η-law for functions: has it survived?

η-law for $A \to B$

Any term $\Gamma \vdash M : A \to B$ can be expanded as

$$\lambda \mathtt{x}.M\mathtt{x}$$

Anything of function type is a function.

This fails in CBV, but it holds in CBN.
Similarly

$$\lambda \mathtt{x}.\ \texttt{error}\ e \quad \simeq_{\mathbf{CBN}} \quad \texttt{error}\ e$$
$$\lambda \mathtt{x}.\ \texttt{print}\ c.\ M \quad \simeq_{\mathbf{CBN}} \quad \texttt{print}\ c.\ \lambda \mathtt{x}.\ M$$

Yet the two sides have different operational behaviour! What's going on?

In CBN, a function gets evaluated only by being applied.

Summary

The pure calculus satisfies all the β- and η-laws.

With computational effects,

- CBV satisfies η for boolean and sum types, but not function types
- CBN satisfies η for function types, but not boolean and sum types.

We want denotational semantics that validate the appropriate η-laws.

We'll do CBV first, as it's easier.

Denotational Semantics of CBV (Moggi)

Take a (strong) monad T on $\mathbf{Set}$.

- For errors: $- + E$
- For printing: $\mathcal{A}^* \times -$

Each type denotes a set (think: semantic domain for terminals)

$$
\begin{aligned}
[\![\texttt{bool}]\!] &= \mathbb{B} \\
[\![A + B]\!] &= [\![A]\!] + [\![B]\!] \\
[\![A \to B]\!] &= [\![A]\!] \to T[\![B]\!]
\end{aligned}
$$

Each term $\Gamma \vdash M : B$ denotes a Kleisli morphism,

i.e. a function $[\![\Gamma]\!] \xrightarrow{[\![M]\!]} T[\![B]\!]$.

To prove the soundness of the denotational semantics, we need a substitution lemma.

Can we obtain $[\![M[N/\mathtt{x}]]\!]$ from $[\![M]\!]$ and $[\![N]\!]$?

CBV Substitution Lemma: What Doesn't Work

Can we obtain $[\![M[N/\mathtt{x}]]\!]$ from $[\![M]\!]$ and $[\![N]\!]$? Not in CBV.

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Example that rules out a general substitution lemma

Define $\mathtt{x} : \mathtt{bool} \vdash M, M' : \mathtt{bool}$ and $\vdash N : \mathtt{bool}$

$$
\begin{aligned}
M &\stackrel{\mathrm{def}}{=} \mathtt{true} \\
M' &\stackrel{\mathrm{def}}{=} \mathtt{case\ x\ of\ \{true.\ true,\ false.\ true\}} \\
N &\stackrel{\mathrm{def}}{=} \mathrm{error\ CRASH} \\
[\![M]\!] &= [\![M']\!] \qquad \text{because } M =_{\eta\,\mathtt{bool}} M' \\
[\![M[N/\mathtt{x}]]\!] &\neq [\![M'[N/\mathtt{x}]]\!]
\end{aligned}
$$

CBV Substitution Lemma: What Doesn't Work

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[\![M]\!] &= [\![M']\!] \qquad \text{because } M =_{\eta\,\mathtt{bool}} M' \\
[\![M[N/\mathtt{x}]]\!] &\neq [\![M'[N/\mathtt{x}]]\!]
\end{aligned}
$$

But we can give a lemma for the substitution of values:

$$
V ::= \ \mathtt{true} \mid \mathtt{false} \mid \mathtt{inl}\ V \mid \mathtt{inr}\ V \mid \lambda\mathtt{x}.M \mid \mathtt{x}
$$

The terminals are the closed values.

Substitution Lemma For Values

Each value $\Gamma \vdash V : B$ denotes a function $\llbracket \Gamma \rrbracket \xrightarrow{\llbracket V \rrbracket^{\mathsf{val}}} \llbracket B \rrbracket$ such that

$$
\begin{array}{ccc}
\llbracket \Gamma \rrbracket & \xrightarrow{\llbracket V \rrbracket^{\mathsf{val}}} & \llbracket B \rrbracket \\
& \searrow_{\llbracket V \rrbracket} & \downarrow^{\eta \llbracket B \rrbracket} \\
& & T\llbracket B \rrbracket
\end{array}
$$

commutes.

Substitution Lemma

Given a term $\Gamma, \mathtt{x} : A \vdash M : B$ and a value $\Gamma \vdash V : A$

we can obtain $\llbracket M[V/\mathtt{x}] \rrbracket$ from $\llbracket M \rrbracket$ and $\llbracket V \rrbracket^{\mathsf{val}}$. It is

$$
\rho \longmapsto \llbracket M \rrbracket (\rho, \mathtt{x} \mapsto \llbracket V \rrbracket^{\mathsf{val}} \rho)
$$

Soundness of CBV Denotational Semantics

Errors

- If $M \Downarrow V$ then $[\![M]\!]\varepsilon = \mathsf{inl}\ ([\![V]\!]^{\mathsf{val}}\varepsilon)$.
- If $M \nDownarrow e$ then $[\![M]\!]\varepsilon = \mathsf{inr}\ e$.

Printing

- If $M \Downarrow m, V$ then $[\![M]\!]\varepsilon = \langle m, [\![V]\!]^{\mathsf{val}}\varepsilon \rangle$.

These are straightforward inductions, using the substitution lemma.

Naive Attempt At CBN: "Carrier" Semantics

Each type denotes a set (think: semantic domain for closed terms).
For example $\texttt{bool} \to (\texttt{bool} \to \texttt{bool})$ should denote $T\mathbb{B} \to (T\mathbb{B} \to T\mathbb{B})$.
We define

$$
\begin{aligned}
\llbracket \texttt{bool} \rrbracket &= T\mathbb{B} \\
\llbracket A + B \rrbracket &= T(\llbracket A \rrbracket + \llbracket B \rrbracket) \\
\llbracket A \to B \rrbracket &= \llbracket A \rrbracket \to \llbracket B \rrbracket
\end{aligned}
$$

Each term $\Gamma \vdash M : B$ should denote a function $\llbracket \Gamma \rrbracket \xrightarrow{\;\llbracket M \rrbracket\;} \llbracket B \rrbracket$.

Carrier Semantics: What Goes Wrong

$$\overline{\Gamma \vdash \mathtt{error}\ e : B}$$

denotes $\rho \mapsto\ ?$

$$\frac{}{\Gamma \vdash \texttt{error } e : B}$$ denotes $\rho \mapsto ?$

Example:

- suppose $B = \texttt{bool} \rightarrow (\texttt{bool} \rightarrow \texttt{bool})$
- then B denotes $(\mathbb{B} + E) \rightarrow ((\mathbb{B} + E) \rightarrow (\mathbb{B} + E))$
- and $\texttt{error } e \simeq_{\textbf{CBN}} \lambda \texttt{x}. \lambda \texttt{y}. \texttt{error } e$
- so the answer should be $\lambda x. \lambda y. \texttt{inr } e$.

Intuition: go down through the function types until we hit a boolean or sum type.

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Intuition: go down through the function types until we hit a boolean or sum type.

A similar problem arises with case.

A CBN type should denote a set X (the carrier) with some designated elements $E \xrightarrow{\text{error}} X$.

This is called an E-pointed set.

Thus `bool` denotes $\mathbb{B} + E$ with $e \mapsto \text{inr } e$.

If $[\![A]\!] = (X, \text{error})$ and $[\![B]\!] = (Y, \text{error}')$,

- then $A + B$ denotes $(X + Y) + E$ with $e \mapsto \text{inr } e$
- and $A \to B$ denotes $X \to Y$ with $e \mapsto \lambda x.\ \text{error}'(e)$.

E-pointed semantics of CBN types

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Can we generalize the notion of E-pointed set to other monads on **Set**?

An *Eilenberg-Moore algebra* for a monad T on **Set** is

- a set X (the carrier)
- a function $TX \xrightarrow{\theta} X$ (the structure)

satisfying

$$X \xrightarrow{\eta X} TX \xleftarrow{\mu X} T^2 X$$

with id from X, θ down from TX, $T\theta$ down from T^2X, to

$$X \xleftarrow{\theta} TX$$

An algebra for the $- + E$ monad is an E-pointed set.

Examples of Algebras

An algebra for the $- + E$ monad is an E-pointed set.

An algebra for $\mathcal{A}^* \times -$ is an $\mathcal{A}$-set

i.e. a set X together with a function $\mathcal{A} \times X \xrightarrow{\ *\ } X$.

This is what we need to interpret

$$\frac{\Gamma \vdash M : B}{\Gamma \vdash \texttt{print } c.\ M : B} \ c \in \mathcal{A}$$

If B denotes $(X, *)$ then $\texttt{print } c.\ M$ denotes $\rho \mapsto c * (\llbracket M \rrbracket \rho)$

3 Ways Of Building Algebras

Free Algebras

Given a set X, the free T-algebra on X has carrier TX and structure μX.

Product Algebras

Given a family of T-algebras (X_i, θ_i), the product algebra $\prod_{i \in I}(X_i, \theta_i)$ has carrier $\prod_{i \in I} X_i$ and structure given pointwise.

Exponential Algebras

Given a set A and a T-algebra (X, θ), the exponential algebra $A \to (X, \theta)$ has carrier $A \to X$ and structure given pointwise.

Algebra Semantics For CBN Types

Let T be a monad on $\mathbf{Set}$.

A type denotes a T-algebra.

- bool denotes the free algebra on $\mathbb{B}$
- If $[\![A]\!] = (X, \theta)$ and $[\![B]\!] = (Y, \phi)$
  - then $A + B$ denotes the free algebra on $X + Y$
  - and $A \rightarrow B$ denotes the exponential algebra $X \rightarrow (Y, \phi)$.

Algebra semantics for CBN terms

Suppose B denotes the algebra (Y, θ).

Then a term $\Gamma \vdash M : B$ denotes a function between the carrier sets $\llbracket \Gamma \rrbracket \xrightarrow{\llbracket M \rrbracket} Y$.

$$\frac{\Gamma \vdash M : \texttt{bool} \quad \Gamma \vdash N : B \quad \Gamma \vdash N' : B}{\Gamma \vdash \texttt{case } M \texttt{ of } \{\texttt{true. } N, \texttt{ false. } N'\} : B}$$

This term denotes

Soundness of algebra semantics for CBN

Errors

- If $M \Downarrow T : B$ then $[\![M]\!]\varepsilon = [\![T]\!]\varepsilon$
- If $M \not\Downarrow e : B$ then $[\![M]\!]\varepsilon = \text{error } e$ where $[\![B]\!] = (X, \text{error})$

Printing

- If $M \Downarrow m, T : B$ then $[\![M]\!]\varepsilon = m \ast\ast([\![T]\!]\varepsilon)$ where $[\![B]\!] = (X, \ast)$

Straightforward inductive proofs using the substitution lemma.

Summary

We have a denotational semantics for errors and printing for CBV and CBN, and shown their correctness.

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A CBV type denotes a set; a CBN type denotes a T-algebra.

Summary

We have a denotational semantics for errors and printing for CBV and CBN, and shown their correctness.

These are instances of a general recipe using a monad T on **Set** and its algebras.

A CBV type denotes a set; a CBN type denotes a T-algebra.

They are fundamentally different things.

We write $F^T X$ for the free T-algebra (TX, μ_X) on X

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and $U^T(X, \theta)$ for the carrier X of a T-algebra (X, θ).

Semantics of Types, Again

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and $U^T(X, \theta)$ for the carrier X of a T-algebra (X, θ).

Our CBN semantics of types can be written

$$
\begin{aligned}
\llbracket \texttt{bool} \rrbracket &= F^T(1+1) \\
\llbracket A + B \rrbracket &= F^T(U^T\llbracket A \rrbracket + U^T\llbracket B \rrbracket) \\
\llbracket A \to B \rrbracket &= U^T\llbracket A \rrbracket \to \llbracket B \rrbracket
\end{aligned}
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$$

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$$
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\end{aligned}
$$

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- computation types which (like CBN types) denote T-algebras.

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value type

$$A ::= U\underline{B} \mid 1 \mid A \times A \mid 0 \mid A + A \mid \sum_{i \in \mathbb{N}} A_i$$

computation type

$$\underline{B} ::= FA \mid A \to \underline{B} \mid 1_\Pi \mid \underline{B} \, \Pi \, \underline{B} \mid \prod_{i \in \mathbb{N}} \underline{B}_i$$

Call-By-Push-Value Types

Call-by-push-value has

- value types which (like CBV types) denote sets
- computation types which (like CBN types) denote T-algebras.

value type $\qquad A ::= \quad U\underline{B} \mid 1 \mid A \times A \mid 0 \mid A + A \mid \sum_{i \in \mathbb{N}} A_i$

computation type $\quad \underline{B} ::= \quad FA \mid A \to \underline{B} \mid 1_{\Pi} \mid \underline{B} \,\Pi\, \underline{B} \mid \prod_{i \in \mathbb{N}} \underline{B}_i$

Strangely function types are computation types, and $\lambda \mathrm{x}.M$ is a computation.

An identifier gets bound to a value, so it has value type.

Judgements

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A context Γ is a finite set of identifiers with associated value type

$$\mathrm{x}_0 : A_0, \ldots, \mathrm{x}_{m-1} : A_{m-1}$$

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Judgement for a value: $\qquad\qquad\qquad \Gamma \vdash^{\mathsf{v}} V : A$

Judgement for a computation: $\qquad\qquad \Gamma \vdash^{\mathsf{c}} M : \underline{B}$

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Judgement for a value: $\qquad\qquad\qquad \Gamma \vdash^{\mathsf{v}} V : A$

Judgement for a computation: $\qquad\quad \Gamma \vdash^{\mathsf{c}} M : \underline{B}$

- A value $\Gamma \vdash^{\mathsf{v}} V : A$ denotes a function $[\![\Gamma]\!] \xrightarrow{[\![V]\!]} [\![A]\!]$
- If $\underline{B}$ denotes (X, θ), then a computation $\Gamma \vdash^{\mathsf{c}} M : \underline{B}$ denotes a function $[\![\Gamma]\!] \xrightarrow{[\![M]\!]} X$.

Note From the viewpoint of monad/algebra semantics, there is no difference between a computation $\Gamma \vdash^{\mathsf{c}} M : \underline{B}$ and a value $\Gamma \vdash^{\mathsf{v}} V : U\underline{B}$.

F and U

The type FA

A computation in FA returns a value in A.

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{c}} \mathtt{return}\ V : FA} \qquad \frac{\Gamma \vdash^{\mathsf{c}} M : FA \quad \Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} N : \underline{B}}{\Gamma \vdash^{\mathsf{c}} M\ \mathtt{to}\ \mathtt{x}.\ N : \underline{B}}$$

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The denotation of $M\ \mathtt{to\ x.}\ N$ uses the structure of $[\![\underline{B}]\!]$.

The type $U\underline{B}$

A value in $U\underline{B}$ is a thunk of a computation in $\underline{B}$.

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{v}} \mathtt{thunk}\ M : U\underline{B}} \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : U\underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{force}\ V : \underline{B}}$$

The constructs $\mathtt{thunk}$ and $\mathtt{force}$ are inverse.
They are invisible in monad/algebra semantics.

An identifier is a value.

$$\frac{}{\Gamma \vdash^{\mathsf{v}} \mathtt{x} : A} \, (\mathtt{x} : A) \in \Gamma \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : A \quad \Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{let} \, V \, \mathtt{be} \, \mathtt{x}. \, M : \underline{B}}$$

We write `let` to bind an identifier.

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A_{\hat{\imath}}}{\Gamma \vdash^{\mathsf{v}} \langle \hat{\imath}, V \rangle : \sum_{i \in I} A_i} \; \hat{\imath} \in I \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : \sum_{i \in I} A_i \quad \Gamma, \mathtt{x} : A_i \vdash^{\mathsf{c}} M_i : \underline{B} \;\; (\forall i \in I)}{\Gamma \vdash^{\mathsf{c}} \mathtt{case} \; V \; \mathtt{of} \; \{\langle i, \mathtt{x} \rangle . M_i\}_{i \in I} : \underline{B}}$$

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A \quad \Gamma \vdash^{\mathsf{v}} V' : A'}{\Gamma \vdash^{\mathsf{v}} \langle V, V' \rangle : A \times A'} \qquad \frac{\Gamma \vdash^{\mathsf{v}} V : A \times A' \quad \Gamma, \mathtt{x} : A, \mathtt{y} : A' \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{case} \; V \; \mathtt{of} \; \langle \mathtt{x}, \mathtt{y} \rangle . M : \underline{B}}$$

The rules for 1 are similar.

Functions

$$\frac{\Gamma, \mathtt{x} : A \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \lambda \mathtt{x}.M : A \to \underline{B}}$$

$$\frac{\Gamma \vdash^{\mathsf{c}} M : A \to \underline{B} \quad \Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{c}} MV : \underline{B}}$$

$$\frac{\Gamma \vdash^{\mathsf{c}} M_i : \underline{B}_i \ (\forall i \in I)}{\Gamma \vdash^{\mathsf{c}} \lambda\{i.M_i\}_{i \in I} : \prod_{i \in I} \underline{B}_i}$$

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \prod_{i \in I} \underline{B}_i}{\Gamma \vdash^{\mathsf{c}} M\hat{\imath} : \underline{B}_{\hat{\imath}}} \ \hat{\imath} \in I$$

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It is often convenient to write applications operand-first, as $V{}^{\backprime}M$ and $\hat{\imath}M$.

The terminals are computations:

$$\mathtt{return}\ V \qquad \lambda \mathtt{x}.M \qquad \lambda \{i.M_i\}_{i \in I}$$

Interpreter

The terminals are computations:

$$\texttt{return } V \qquad \lambda\texttt{x}.M \qquad \lambda\{i.M_i\}_{i \in I}$$

To evaluate

- $\texttt{return } V$: return $\texttt{return } V$.
- $M \texttt{ to x. } N$: evaluate M. If it returns $\texttt{return } V$, then evaluate $N[V/\texttt{x}]$.
- $\lambda\texttt{x}.N$: return $\lambda\texttt{x}.N$.
- MV: evaluate M. If it returns $\lambda\texttt{x}.N$, evaluate $N[V/\texttt{x}]$.
- $\lambda\{i.N_i\}_{i \in I}$: return $\lambda\{i.N_i\}_{i \in I}$.
- $M\hat{\imath}$: evaluate M. If it returns $\lambda\{i.N_i\}_{i \in I}$, evaluate $N_{\hat{\imath}}$.
- $\texttt{let } V \texttt{ be x. } M$: evaluate $M[V/\texttt{x}]$.
- $\texttt{force thunk } M$: evaluate M.
- $\texttt{case } \langle\hat{\imath}, V\rangle \texttt{ of } \{\langle i, \texttt{x}\rangle.M_i\}_{i \in I}$: evaluate $M_{\hat{\imath}}[V/\texttt{x}]$.
- $\texttt{case } \langle V, V'\rangle \texttt{ of } \langle\texttt{x}, \texttt{y}\rangle.M$: evaluate $M[V/\texttt{x}, V'/\texttt{y}]$.

A CBV type translates into a value type.

$$A \to B \quad \mapsto \quad U(A \to FB)$$

A CBV term $\mathtt{x} : A, \mathtt{y} : B \vdash M : C$ translates as $\mathtt{x} : A, \mathtt{y} : B \vdash^{\mathtt{c}} M : FC$.

$$
\begin{aligned}
\mathtt{x} \quad &\mapsto \quad \mathtt{return\ x} \\
\lambda\mathtt{x}.\ M \quad &\mapsto \quad \mathtt{return\ thunk}\ \lambda\mathtt{x}.\ M \\
M\ N \quad &\mapsto \quad M\ \mathtt{to\ f}.\ N\ \mathtt{to\ y}.\ ((\mathtt{force\ f})\ \mathtt{y}) \\
\mathtt{let}\ M\ \mathtt{be\ x}.\ N \quad &\mapsto \quad M\ \mathtt{to\ y}.\ \mathtt{let\ y\ be\ x}.\ N
\end{aligned}
$$

Decomposing CBV into CBPV

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\text{or} \quad &\mapsto \quad M\ \mathtt{to\ x}.\ N
\end{aligned}
$$

Decomposing CBN into CBPV

A CBN type translates into a computation type.

$$\begin{aligned}
\texttt{bool} &\mapsto F(1+1) \\
\underline{A} + \underline{B} &\mapsto F(U\underline{A} + U\underline{B}) \\
\underline{A} \to \underline{B} &\mapsto U\underline{A} \to \underline{B}
\end{aligned}$$

A CBN term $\texttt{x} : \underline{A}, \texttt{y} : \underline{B} \vdash M : \underline{C}$ translates as $\texttt{x} : U\underline{A}, \texttt{y} : U\underline{B} \vdash^{\texttt{c}} M : \underline{C}$.

$$\begin{aligned}
\texttt{x} &\mapsto \texttt{force x} \\
\texttt{let } M \texttt{ be x}. \, N &\mapsto \texttt{let (thunk } M) \texttt{ be x}. \, N \\
\lambda \texttt{x}. \, M &\mapsto \lambda \texttt{x}. \, M \\
M \, N &\mapsto M \, (\texttt{thunk } N) \\
\texttt{inl } M &\mapsto \texttt{return inl thunk } M
\end{aligned}$$

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Summary

We've seen the call-by-push-value calculus, its operational and monad/algebra semantics.

The translations from CBV and CBN into CBPV preserve these semantics.

Moggi's TA is UFA.

But

- the monad/algebra semantics makes thunk and force invisible
- we still don't understand why a function is a "computation".

Antecedents: CBV

- Landin's ISWIM: CBV λ-calculus with effects.
  Influenced ML and other languages.
- Plotkin: λ_v-calculus provided equations for CBV with divergence.
- Numerous researchers: CPS transforms for CBV.
- Felleisen *et al*: CBV semantics for various effects.
- Moggi: λ_c-calculus, monads for CBV and monadic metalanguage.
- Power and Robinson: Freyd categories for CBV.
- Benton and Kennedy: MIL-lite.
- Thielecke: `thunk` and `force` constructs in CBV with `callcc`.

- Plotkin: CPS transform for CBN without η-law.
- Abramsky and Ong: untyped lazy λ-calculus.
- Ong: typed lazy λ-calculus.
- Moggi: monads for CBN without η-law.

- Plotkin's PCF, a CBN calculus for recursion.
- Hennessy and Ashcroft: recursion and nondeterminism.
- O'Hearn: semantics of conditional in Reynolds' Idealized Algol.
- Streicher and Reus: semantics of control effects in CBN.
- $\neg\neg$ translations of classical logic. Also CBV.
- Game semantics. Also CBV.

Calculi combining CBV and CBN

- Various calculi based on CPS.
- Filinski's Effect-PCF provided the CBPV `to` construct. U is implicit.
- Howard's $\lambda^{\mu\nu\perp}$ calculus for recursion. U is implicit.
- Egger, Møgelberg and Simpson's Effect Calculus emphasizes connection to Benton's Linear/Nonlinear Logic. U is implicit.
- Marz' SFPL for recursion and sequentiality. F is implicit.
- Ehrhard's variant of call-by-push-value, similar to SFPL. F is implicit.
- Nygaard and Winskel's HOPLA for recursion and nondeterminism. F is implicit.
- Laurent's LLP has extra type constructors not included in CBPV.
- Harper, Licata and Zeilberger's Polarized Intuitionistic Logic.

CK-machine

An operational semantics due to Felleisen and Friedman (1986).
And Landin, Krivine, Streicher and Reus, Bierman, Pitts, ...

It is suitable for sequential languages whether CBV, CBN or CBPV.

At any time, there's a computation (C) and a stack of contexts (K).

Initially, K is empty.

Some authors make K into a single context, called an "evaluation context".

To evaluate M to x. N: evaluate M. If it returns return V, then evaluate $N[V/\mathtt{x}]$.

M to x. N	K	$\leadsto$
M	to x. $N :: K$	

return V	to x. $N :: K$	$\leadsto$
$N[V/\mathtt{x}]$	K	

To evaluate $V`M$: evaluate M. If it returns $\lambda\mathtt{x}.N$, evaluate $N[V/\mathtt{x}]$.

$V`M$	K	$\leadsto$
M	$V :: K$	

$\lambda\mathtt{x}.N$	$V :: K$	$\leadsto$
$N[V/\mathtt{x}]$	K	

Those function rules again

$V`M$	K	$\rightsquigarrow$
M	$V :: K$	

$\lambda\mathtt{x}.N$	$V :: K$	$\rightsquigarrow$
$N[V/\mathtt{x}]$	K	

Those function rules again

$$
\begin{array}{|lll|}
\hline
V`M & K & \rightsquigarrow \\
M & V :: K & \\
\hline
\end{array}
$$

$$
\begin{array}{|lll|}
\hline
\lambda\mathtt{x}.N & V :: K & \rightsquigarrow \\
N[V/\mathtt{x}] & K & \\
\hline
\end{array}
$$

We can read $V`$ as an instruction "push V".

We can read $\lambda\mathtt{x}$ as an instruction "pop x".

Those function rules again

$V`M$	K	$\rightsquigarrow$
M	$V :: K$	

$\lambda\mathrm{x}.N$	$V :: K$	$\rightsquigarrow$
$N[V/\mathrm{x}]$	K	

We can read $V`$ as an instruction "push V".

We can read $\lambda\mathrm{x}$ as an instruction "pop x".

Revisiting some equations:

$$
\begin{aligned}
V ` \lambda\mathrm{x}.\, M &= M[V/\mathrm{x}] \\
M &= \lambda\mathrm{x}.\ \mathrm{x} ` M \qquad \text{(x fresh)} \\
\lambda\mathrm{x}.\ \mathtt{error}\ e &= \mathtt{error}\ e \\
\lambda\mathrm{x}.\ \mathtt{print}\ c.\, M &= \mathtt{print}\ c.\ \lambda\mathrm{x}.M
\end{aligned}
$$

A value is, a computation does.

- A value of type $U\underline{B}$ is a thunk of a computation of type $\underline{B}$.
- A value of type $\sum_{i\in I}A_i$ is a pair $\langle i, V\rangle$.
- A value of type $A \times A'$ is a pair $\langle V, V'\rangle$.

- A computation of type FA returns a value of type A.
- A computation of type $A \to \underline{B}$
  pops a value of type A
  then behaves in $\underline{B}$.
- A computation of type $\prod_{i\in I}\underline{B}_i$
  pops a tag $i \in I$
  then behaves in $\underline{B}_i$.

What's in a stack?

A stack consists of

- arguments that are values
- arguments that are tags
- frames taking the form `to x.` N.

Example program of type F nat

```
print "hello0".
let 3 be x.
let  thunk (
        print "hello1".
        λz.
        print "we just popped "z.
        return x + z
     ) be y.
print "hello2".
( print "hello3".
  7`
  print "we just pushed 7".
  force y
) to w.
print "w is bound to " + w.
return w + 5
```

Typing the CK-machine

Initial configuration to evaluate $\Gamma \vdash^c P \colon \underline{C}$

Γ	P	$\underline{C}$	`nil`	$\underline{C}$

Transitions

Γ	M `to x.` N	$\underline{B}$	K	$\underline{C}$	$\rightsquigarrow$
Γ	M	FA	`to x.` $N :: K$	$\underline{C}$	

Γ	`return` V	FA	`to x.` $N :: K$	$\underline{C}$	$\rightsquigarrow$
Γ	$N[V/\text{x}]$	$\underline{B}$	K	$\underline{C}$	

Typically Γ would be empty and $\underline{C} = F\,\texttt{bool}$.

Typing the CK-machine

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Transitions

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Γ	M	FA	`to x.` $N :: K$	$\underline{C}$	

Γ	`return` V	FA	`to x.` $N :: K$	$\underline{C}$	$\leadsto$
Γ	$N[V/x]$	$\underline{B}$	K	$\underline{C}$	

Typically Γ would be empty and $\underline{C} = F\,\texttt{bool}$. We write

$\Gamma \vdash^k K : \underline{B} \Longrightarrow \underline{C}$ to mean that K can accompany a computation of type $\underline{B}$ during evaluation.

Typing a stack

$$\overline{\Gamma \vdash^\mathsf{k} \mathtt{nil} : \underline{C} \Longrightarrow \underline{C}}$$

$$\frac{\Gamma, \mathtt{x} : A \vdash^\mathsf{c} M : \underline{B} \qquad \Gamma \vdash^\mathsf{k} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^\mathsf{k} \mathtt{to\ x.}\ M :: K : FA \Longrightarrow \underline{C}}$$

$$\frac{\Gamma \vdash^\mathsf{k} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^\mathsf{k} \hat{\imath} :: K : \prod_{i \in I} \underline{B}_i \Longrightarrow \underline{C}}\ \hat{\imath} \in I$$

$$\frac{\Gamma \vdash^\mathsf{v} V : A \qquad \Gamma \vdash^\mathsf{k} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^\mathsf{k} V :: K : A \to \underline{B} \Longrightarrow \underline{C}}$$

Typing rules, read off from the CK-machine

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$$\frac{\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^{\mathsf{k}} \hat{\imath} :: K : \prod_{i \in I} \underline{B}_i \Longrightarrow \underline{C}} \ \hat{\imath} \in I$$

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Typing a CK-configuration

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B} \qquad \Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}}{\Gamma \vdash^{\mathsf{ck}} (M, K) : \underline{C}}$$

Continuations

A continuation is a stack from an F type. For example:

$$\Gamma \vdash^{\mathsf{k}} \texttt{to x.}\ M :: K : FA \Longrightarrow \underline{C}$$

It describes what happens next once it receives a value.

Special Stacks

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Top-Level Stack

The top-level stack is

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and $\underline{C}$ is the top-level type.
If $\underline{C}$ is an F type, then $\texttt{nil}$ is the top-level continuation: it receives a value and returns it to the user.

Denotational semantics of stacks

Suppose $\llbracket \underline{C} \rrbracket = (Y, \phi)$. The behaviour of $\Gamma \vdash^{\mathsf{ck}} (M, K) : \underline{C}$ depends on the environment:

$$\llbracket \Gamma \rrbracket \xrightarrow{\ \llbracket (M,K) \rrbracket\ } Y$$

to be preserved by each transition.

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Suppose also $[\![\underline{B}]\!] = (X, \theta)$. A stack $\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}$ transforms computations to CK-configurations. So we get a function

$$[\![\Gamma]\!] \times X \xrightarrow{[\![K]\!]} Y$$

homomorphic in its second argument.

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because if M raises an error or prints, then so does M, K.

We assume there's no exception handling.

We have an adjunction between the category of values (sets and functions) and the category of stacks (T-algebras and homomorphisms).

$$\mathbf{Set} \xrightarrow[\underset{U^T}{\perp}]{F^T} \mathbf{Set}^T$$

This resolves the monad T on $\mathbf{Set}$.

Consider CBPV extended with two storage cells:
1 stores a natural number, and $1'$ stores a boolean.

State

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$\mathtt{l}$ stores a natural number, and $\mathtt{l}'$ stores a boolean.

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{c}} \mathtt{l} := n.\ M : \underline{B}} \ n \in \mathbb{N} \qquad \frac{\Gamma \vdash^{\mathsf{c}} M_n : \underline{B} \ \ (\forall n \in \mathbb{N})}{\Gamma \vdash^{\mathsf{c}} \mathtt{read\ l\ as}\ \{n.\ M_n\}_{n \in \mathbb{N}} : \underline{B}}$$

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A state is $\mathtt{l} \mapsto n, \mathtt{l}' \mapsto b$.

The set of states is $S \cong \mathbb{N} \times \mathbb{B}$.

Big-step semantics for state

The big-step semantics takes the form $s, M \Downarrow s', T$.

A pair s, M is called an SC-configuration.

We can type these using

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B}}{\Gamma \vdash^{\mathsf{sc}} (s, M) : \underline{B}} \; s \in S$$

Monad/algebra semantics for state

Moggi's monad for global state is $S \to (S \times -)$.

We can take algebras for this and obtain a denotational semantics of CBPV with state.

Monad/algebra semantics for state

Moggi's monad for global state is $S \to (S \times -)$.

We can take algebras for this and obtain a denotational semantics of CBPV with state.

But it doesn't fit well with SC-configurations.

We'd like a soundness result of the following form:

$$\text{If } s, M \Downarrow s', T \text{ then } [\![s, M]\!]\varepsilon = [\![s', T]\!]\varepsilon$$

This requires an SC-configuration to have a denotation.

Semantics of SC-configurations

Denotation of A is a semantic domain for values of type A. Like in monad semantics.

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The behaviour of an SC-configuration $\Gamma \vdash^{\mathsf{sc}} (s, M) : \underline{B}$ depends on the environment:

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The behaviour of a computation $\Gamma \vdash^{\mathsf{c}} M : \underline{B}$ depends on the state and environment:

$$S \times \llbracket \Gamma \rrbracket \xrightarrow{\ \llbracket M \rrbracket\ } \llbracket \underline{B} \rrbracket$$

State: semantics of types

An SC-configuration of type FA will terminate as $s, \mathtt{return}\ V$.

$$\llbracket FA \rrbracket = S \times \llbracket A \rrbracket$$

An SC-configuration of type $A \to \underline{B}$ will pop $\mathtt{x} : A$, then behave in $\underline{B}$.

$$\llbracket A \to \underline{B} \rrbracket = \llbracket A \rrbracket \to \llbracket \underline{B} \rrbracket$$

An SC-configuration of type $\prod_{i \in I} \underline{B}_i$ will pop $i \in I$, then behave in $\underline{B}_i$.

$$\llbracket \prod_{i \in I} \underline{B}_i \rrbracket = \prod_{i \in I} \llbracket \underline{B}_i \rrbracket$$

A value $\Gamma \vdash^{\mathsf{v}} V : U\underline{B}$ can be forced in any state s, giving an SC-configuration $s, \mathtt{force}\ V$.

$$\llbracket U\underline{B} \rrbracket = S \to \llbracket \underline{B} \rrbracket$$

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An SC-configuration of type $A \to \underline{B}$ will pop $\mathtt{x} : A$, then behave in $\underline{B}$.

$$\llbracket A \to \underline{B} \rrbracket = \llbracket A \rrbracket \to \llbracket \underline{B} \rrbracket$$

An SC-configuration of type $\prod_{i \in I} \underline{B}_i$ will pop $i \in I$, then behave in $\underline{B}_i$.

$$\llbracket \textstyle\prod_{i \in I} \underline{B}_i \rrbracket = \textstyle\prod_{i \in I} \llbracket \underline{B}_i \rrbracket$$

A value $\Gamma \vdash^{\mathsf{v}} V : U\underline{B}$ can be forced in any state s, giving an SC-configuration $s, \mathtt{force}\ V$.

$$\llbracket U\underline{B} \rrbracket = S \to \llbracket \underline{B} \rrbracket$$

We recover standard semantics for CBV,
and O'Hearn's semantics for CBN.

The SCK-machine

We replace $\vdash^{\mathsf{ck}}$ with $\vdash^{\mathsf{sck}}$.

$$\frac{\Gamma \vdash^{\mathsf{c}} M : \underline{B} \qquad \Gamma \,|\, \underline{B} \vdash^{\mathsf{k}} K : \underline{C}}{\Gamma \vdash^{\mathsf{sck}} (s, M, K) : \underline{C}} \; s \in S$$

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The behaviour of an SCK-configuration $\Gamma \vdash^{\mathsf{sck}} (s, M, K) : \underline{C}$ depends on the environment:

$$[\![\Gamma]\!] \xrightarrow{\;[\![(s,M,K)]\!]\;} [\![\underline{C}]\!]$$

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A stack $\Gamma \vdash^{\mathsf{k}} K : \underline{B} \Longrightarrow \underline{C}$ transforms SC-configuration behaviours to SCK-configuration behaviours:

$$[\![\Gamma]\!] \times [\![\underline{B}]\!] \xrightarrow{\;[\![K]\!]\;} [\![\underline{C}]\!]$$

State: the value/stack adjunction

We've seen that a stack $\vdash^k K : \underline{B} \Longrightarrow \underline{C}$ denotes a function
$\llbracket \underline{B} \rrbracket \xrightarrow{\llbracket K \rrbracket} \llbracket \underline{C} \rrbracket$.

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We have an adjunction

$$\mathbf{Set} \; \underset{S \to -}{\overset{S \times -}{\rightleftarrows}} \; \mathbf{Set}$$

between values and stacks.

Catching and throwing a stack

We extend CBPV with Crolard's instructions for changing the stack.

- `catch` α means "let α be the current stack".
- `throw` K means "change the current stack to K".

Γ	`catch` α. M	$\underline{B}$	K	$\underline{C} \mid \Delta$	$\rightsquigarrow$
Γ	$M[K/\alpha]$	$\underline{B}$	K	$\underline{C} \mid \Delta$	

Γ	`throw` K. M	$\underline{B}'$	K'	$\underline{C} \mid \Delta$	$\rightsquigarrow$
Γ	M	$\underline{B}$	K	$\underline{C} \mid \Delta$	

Typing judgements for control

The stack context Δ consists of stack names $\alpha : \underline{B}$.

So we have typing judgements:

$$\Gamma \vdash^{\mathsf{v}} V : A \mid \Delta \qquad \Gamma \vdash^{\mathsf{c}} M : \underline{B} \mid \Delta$$

During execution, the top-level type $\underline{C}$ must be indicated:

$$\Gamma \vdash^{\mathsf{v}} V : A \; [\underline{C}] \; \Delta \qquad \Gamma \vdash^{\mathsf{c}} M : \underline{B} \; [\underline{C}] \; \Delta$$
$$\Gamma \vdash^{\mathsf{k}} K : \underline{B} \implies \underline{C} \mid \Delta \qquad \Gamma \vdash^{\mathsf{ck}} (M, K) : \underline{C} \mid \Delta$$

Typically Γ and Δ would be empty and $\underline{C} = F\,\texttt{bool}$.

Fix a set R, the semantic domain for CK-configurations.

Monad/algebra semantics of control

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Maybe we can use algebras for this to build a denotational semantics of control.

Semantics of control using stacks

The denotation of $\underline{B}$ is a semantic domain for stacks from $\underline{B}$.

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The behaviour of a computation $\Gamma \vdash^c M : \underline{B}\ [\underline{C}]\ \Delta$ depends on the environment, current stack, top-level stack and stack environment:

$$[\![\Gamma]\!] \times [\![\underline{B}]\!] \times [\![\underline{C}]\!] \times [\![\Delta]\!] \xrightarrow{[\![M]\!]} R$$

A value $\Gamma \vdash^v V : A\ [\underline{C}]\ \Delta$ denotes

$$[\![\Gamma]\!] \times [\![\underline{C}]\!] \times [\![\Delta]\!] \xrightarrow{[\![V]\!]} [\![A]\!]$$

A stack $\Gamma \vdash^k K : \underline{B} \Longrightarrow \underline{C}\ |\ \Delta$ denotes

$$[\![\Gamma]\!] \times [\![\underline{C}]\!] \times [\![\Delta]\!] \xrightarrow{[\![K]\!]} [\![\underline{B}]\!]$$

A CK-configuration $\Gamma \vdash^{ck} (M, K) : \underline{C}\ |\ \Delta$ denotes

$$[\![\Gamma]\!] \times [\![\underline{C}]\!] \times [\![\Delta]\!] \xrightarrow{[\![(M,K)]\!]} R$$

to be preserved by each transition.

Control: semantics of types

A stack from FA receives a value $\mathtt{x} : A$ and then behaves as a configuration.

$$[\![FA]\!] = [\![A]\!] \to R$$

A stack from $A \to \underline{B}$ is a pair $V :: K$.

$$[\![A \to \underline{B}]\!] = [\![A]\!] \times [\![\underline{B}]\!]$$

A stack from $\prod_{i \in I} \underline{B}_i$ is a pair $i :: K$.

$$[\![\textstyle\prod_{i \in I} \underline{B}_i]\!] = \textstyle\sum_{i \in I} [\![\underline{B}_i]\!]$$

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We recover standard continuation semantics for CBV,
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A stack

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denotes a function

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So we have an adjunction

$$\mathbf{Set} \underset{-\to R}{\overset{-\to R}{\underset{\perp}{\rightleftarrows}}} \mathbf{Set}^{\mathrm{op}}$$

between values and stacks with top-level type.

Summary of models

For every monad T on **Set** we have an adjunction

$$\mathbf{Set} \xrightarrow[\underset{U^T}{\perp}]{F^T} \mathbf{Set}^T$$

This is useful for modelling CBPV with errors and printing.

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