Final coalgebras from corecursive algebras

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Outline

Transition systems

Let $\mathcal{A}$ be a set of labels.
An image-countable $\mathcal{A}$-labelled transition system consists of

- a set X
- a function $X \to (\mathcal{P}_c X)^{\mathcal{A}}$

This is a coalgebra for the endofunctor on **Set**

$$B \,:\, X \mapsto (\mathcal{P}_c X)^{\mathcal{A}}$$

How can we construct a final coalgebra?

Strongly extensional quotient of an all-encompassing coalgebra

Let P be an all-encompassing B-coalgebra:
every element of every B-coalgebra is bisimilar to some element of P.

Then the strongly extensional quotient (quotient by bisimilarity) of P is a final coalgebra.

Examples of all-encompassing coalgebras, for $\mathcal{A} = 1$

- (Large) The sum of all coalgebras.
- The sum of all coalgebras carried by a subset of $\mathbb{N}$.
- The set of non-well-founded terms for a constant and an ω-ary operation.

Hennessy-Milner logic

With countable conjunctions, non-bisimilar states can be distinguished.

$$\phi ::= \bigwedge_{i \in I} \phi_i \mid \neg\phi \mid [a]\phi \quad (I \text{ countable})$$

It's sufficient to take the $\Diamond$-layered formulas.

$$\phi ::= \langle a \rangle \, (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg\phi_j)$$

Semantics in a colagebra (X, ζ)

$$u \models \langle a \rangle \, (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg\phi_j)$$
$$\iff$$
$$\exists x \in (\zeta(u))_a. \, (\forall i \in I. x \models \phi_i \wedge \forall j \in J. \, x \not\models \psi_j)$$

Formulas and states

For a state x, write $(\!|x|\!) = \{\phi \mid x \models \phi\}$.

For a formula ϕ, write $[\![\phi]\!]_{X,\zeta} = \{x \in X \mid x \models \phi\}$.

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Theorem

$x \simeq y$ iff $(\!|x|\!) = (\!|y|\!)$

$(\Leftarrow)$ is soundness.

$(\Rightarrow)$ is expressivity.

Final coalgebra from modal logic

Theorem

$x \sim y$ iff $(\!|x|\!) = (\!|y|\!)$

Gives a final coalgebra whose states are sets of formulas.

Take $\{(\!|x|\!) \mid (X, \zeta) \text{ a } T\text{-coalgebra}, x \in X\}$.

The structure at $(\!|x|\!)$ applies $X \xrightarrow{\zeta} FX \xrightarrow{F(\!|-|\!)} FM$

(Goldblatt; Kupke and Leal)

The Problem

$\{[\![x]\!]_{X,\zeta} \mid (X,\zeta) \text{ a } T\text{-coalgebra}, x \in X\}$

This is very similar to quotienting by bisimilarity.

It is constructed out of general coalgebras.

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Our question

Can we build a final coalgebra purely from the logic,
without reference to other coalgebras?

We need to say when a set of formulas is of the form $[\![x]\!]_{X,\zeta}$.

The image-finite case

The functor is $B : X \mapsto (\mathcal{P}^{\mathsf{f}} X)^{\mathcal{A}}$.

Build the canonical model, consisting of sets of formulas deductively closed in the modal logic K.

This is a transition system.

The hereditarily image-finite elements form a final coalgebra.

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But what about the image-countable case?

Starting-point: a B-algebra

The carrier is the set Form of theories, i.e. sets of $\Diamond$-layered formulas.

The structure $\alpha : B\,\mathrm{Form} \to \mathrm{Form}$ is given as follows.

For $\mathcal{M} \in B\,\mathrm{Form}$, the formula $\langle a \rangle \left(\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg \psi_j\right)$ is in $\alpha\mathcal{M}$

when there exists $M \in \mathcal{M}a$ such that $\forall i \in I.\ \phi_i \in M$ and $\forall j \in J.\, \psi_j \notin M$.

Think of $\mathcal{M}$ as describing the semantics of the successors of a node x, then $\alpha\mathcal{M}$ is the semantics of x.

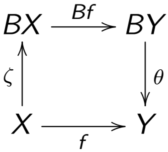
Properties of the B-algebra

The B-algebra we have just seen is

- corecursive
- injectively structured.

Corecursive algebra

A map from a B-coalgebra to a B-algebra



Think: to recursively define $f(x)$, first parse x into parts, apply f to each part, then combine the results.

Corecursive algebra

A map from a B-coalgebra to a B-algebra

$$\begin{array}{ccc} BX & \xrightarrow{Bf} & BY \\ {\scriptstyle\zeta}\uparrow & & \downarrow{\scriptstyle\theta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Think: to recursively define $f(x)$, first parse x into parts, apply f to each part, then combine the results.

A coalgebra is recursive when there's a unique map to every algebra. Corresponds to well-foundedness. (Taylor)

An algebra is corecursive when there's a unique map from every coalgebra. Our algebra of fomulas sets is corecursive.

Co-founded elements of an algebra

Let S be a signature, i.e. a set of operations each with an arity.

Let $(Y, \ldots)$ be an S-algebra.

An element of Y is co-founded when
it is of the form $c(y_i \mid i \in \mathsf{ar}(c))$
with each y_i co-founded.

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We shall generalize this to B-coalgebras

where B is an endofunctor on **Set** preserving injections.

The co-founded part of an algebra

Starting with a B-algebra (Y, θ), we define a monotone endofunction p on $\mathcal{P}Y$.

For $U \in \mathcal{P}Y$ with inclusion $i_U : U \to Y$, we have

$$\begin{array}{ccc} BU & \xrightarrow{Bi_U} & BY \\ {\scriptstyle r_U}\downarrow & & \downarrow{\scriptstyle \theta} \\ p(U) & \xrightarrow[i_{p(U)}]{} & Y \end{array}$$

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This is a monotone endofunction on $\mathcal{P}Y$.

A prefixpoint of p is a subalgebra of (Y, θ).

The greatest postfixpoint νp is called the co-founded part of (Y, θ).

It is a surjectively structured algebra, in fact the coreflection of (Y, θ) into surjectively structured algebras.

Facts about the co-founded part

Claim The (co-founded part)$^{-1}$ of our algebra is a final coalgebra, and the least subalgebra is an initial algebra.

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- The co-founded part of a corecursive algebra (Y, θ) is corecursive.
- If (Y, θ) is injectively structured, the co-founded part is injectively and surjectively structured, hence bijectively structured.
- Any isomorphically structured corecursive algebra gives us a final coalgebra.
- If (Y, θ) is injectively structured, then its least subalgebra is an initial algebra. (Adámek and Trnková)

The recipe

Let B be an endofunctor on **Set** preserving injections.

Take an injectively structured, corecursive B-algebra.

Its (co-founded part)$^{-1}$ is a final B-coalgebra,

and its least subalgebra is an initial B-algebra.

Modal logics in general

We can improve and generalize this recipe

using Klin's framework of expressive modal logic on a dual adjunction.

Adjunctions and bimodules

What is an adjunction between $\mathcal{C}$ and $\mathcal{D}^{\mathrm{op}}$?

Definition of dual adjunction

Functors $\mathcal{O}_* : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $\mathcal{O}^* : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, and

$$\mathcal{C}(X, \mathcal{O}^*\Phi) \cong \mathcal{D}(\Phi, \mathcal{O}_*X) \qquad \text{natural in } X \in \mathcal{C}^{\mathrm{op}}, \Phi \in \mathcal{D}.$$

Alternative definition of dual adjunction

A functor $\mathcal{O} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ (aka bimodule, profunctor), and

$$\mathcal{C}(X, \mathcal{O}^*\Phi) \cong \mathcal{O}(X, \Phi) \cong \mathcal{D}(\Phi, \mathcal{O}_*X) \qquad \text{natural in } X \in \mathcal{C}^{\mathrm{op}}, \Phi \in \mathcal{D}.$$

Dual adjunction for satisfaction relations

Consider this dual adjunction between **Set** and **Set**.

$$\mathbf{Set}(X, \mathcal{P}\Phi) \;\cong\; \mathbf{Rel}(X, \Phi) \;\cong\; \mathbf{Set}(\Phi, \mathcal{P}X)$$

Suppose X carries a coalgebra and Φ is the set of formulas.

$$(\!|-|\!) \;\leftrightarrow \models \leftrightarrow\; [\![-]\!]$$

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Intuitions

- An object $X \in \mathcal{C}$ is a set of states.
- An object $\Phi \in \mathcal{D}$ is a set of formulas.
- $\mathcal{O}(X, \Phi)$ is the set of satisfaction relations.
- $\mathcal{O}_* X$ is the set of predicates on X.
- $\mathcal{O}^* \Phi$ is the set of theories of Φ.

Syntax of a modal logic

The syntax is represented by an endofunctor L on $\mathcal{D}$.

$L\Phi$ is the set of single-layer formulas with atoms in Φ.

Example: $\Diamond$-layered formulas

$\mathcal{D}$ is **Set**.

$L\Phi$ is the set of formulas

$$\langle a \rangle \, (\bigwedge_{i \in I} \phi_i \wedge \bigwedge_{j \in J} \neg \psi_j) \qquad\qquad (\phi_i, \psi_j \in \Phi)$$

More concisely $L\Phi = \mathcal{A} \times \mathcal{P}_c\Phi \times \mathcal{P}_c\Phi$.

The set of formulas form an initial L-algebra.

Semantics of a modal logic

$L\Phi$ is the set of single-layer formula with atoms in Φ.

BX is the set of single-step behaviours ending in a state in X.

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The semantics is given by a map

$$\rho_{X,\Phi} \;:\; \mathcal{O}(X,\Phi) \to \mathcal{O}(BX, L\Phi) \text{ natural in } X \in \mathcal{C}^{\mathrm{op}}, \Phi \in \mathcal{D}^{\mathrm{op}}$$

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Example: $\Diamond$-layered formulas

$$s(\rho_{X,\Phi}(\models))\langle a\rangle \ (\textstyle\bigwedge_{i\in I}\phi_i \wedge \bigwedge_{j\in J}\neg\psi_j)$$

$$\Longleftrightarrow$$

$$\exists x \in s_a.\ (\forall i \in I. x \models \phi_i \wedge \forall j \in J.\ x \not\models \psi_j)$$

Putting it together

Given an endofunctor B on $\mathcal{C}$, a modal logic consists of

- a dual adjunction $(\mathcal{D}, \mathcal{O})$ to $\mathcal{C}$
- (syntax) an endofunctor L on $\mathcal{D}$
- (semantics) a natural transformation $\rho_{X,\Phi} : \mathcal{O}(X, \Phi) \to \mathcal{O}(BX, L\Phi)$

Mates of the semantics

The semantics can be expressed in terms of $\mathcal{O}_*$:

$$\rho_*^X \ : \ L\mathcal{O}_*X \to \mathcal{O}_*BX$$

And it can be expressed in terms of $\mathcal{O}^*$:

$$\rho_\Phi^* \ : \ B\mathcal{O}^*\Phi \to \mathcal{O}^*L\Phi$$

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Expressiveness (Klin)

Suppose $\mathcal{C} = \mathbf{Set}$, and B preserves injections.

The modal logic is expressive when ρ_Φ^* is injective for all Φ.

Improved recipe

Let B be an endofunctor on **Set** preserving injections.

Let $(\mathcal{D}, \mathcal{O}, L, \rho)$ be an expressive modal logic, with an initial L-algebra. Then the B-algebra

$$B\mathcal{O}^*\mu L \to \mathcal{O}^* L\mu L \cong \mathcal{O}^*\mu L$$

is corecursive and injectively structured.

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Let $(\mathcal{D}, \mathcal{O}, L, \rho)$ be an expressive modal logic, with an initial L-algebra. Then the B-algebra

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is corecursive and injectively structured.

So its (coinductive part)$^{-1}$ is a final B-coalgebra

and its least subalgebra is an initial B-algebra.

In the paper

Generalizing from **Set** to other categories with a suitable factorization system

e.g. **Poset** and $\mathbf{Set}^{\mathrm{op}}$.

Conclusion

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If B preserves arbitrary intersections, it's ω.