Locally graded categories

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Outline

Things and sets

- Ordinary mathematics is about things.
- A collection of things is itself a thing, provided its size is limited. Such a collection is called a set.
- Categorical mathematics requires arbitrary collections of things. These are called classes.
- Sometimes, classes are not enough, so we need a bigger ontology.

k-classes and k-entities

- A 0-class is a set.
- A 0-entity is a thing.
- A 1-class is a collection of 0-entities, i.e. a class.
- We shall not define 1-entity but assume the following.
  - Every thing is a 1-entity.
  - Every class is a 1-entity.
  - Every ordered pair of 1-entities is a 1-entity.
  - Every class-indexed tuple of 1-entities is a 1-entity.
- A 2-class is a collection of 1-entities.
- Likewise k-entity and k-class for $k \in \mathbb{N}$.

Modelling k-entities and k-classes

Here's a way to interpret our terminology. There are others.
We work in ZFC, perhaps with urelements,
and one Grothendieck universe parameter $\mathfrak{U}$.

- Thing → element of $\mathfrak{U}$.
- Set → set in $\mathfrak{U}$.
- Class → subset of $\mathfrak{U}$.
- "1-entity" is inductively defined by the axioms.
- 2-class → set of 1-entities.
- Etc.

Application: sizes of categories

A category $\mathcal{C}$ is

- small when ob $\mathcal{C}$ and each $\mathcal{C}(a, b)$ is a set
- moderate when ob $\mathcal{C}$ and each $\mathcal{C}(a, b)$ is a class.
- 2-moderate when ob $\mathcal{C}$ and each $\mathcal{C}(a, b)$ is a 2-class.
- light when ob $\mathcal{C}$ is a class and each $\mathcal{C}(a, b)$ is a set.

Light = moderate + locally small.

Examples

- The category of natural numbers and functions is small.
- $\mathbf{Set}$ is light but not small.
- The category of sets and multirelations is moderate but not light.
- The category $[\mathbf{Set}, \mathbf{Set}]$ is 2-moderate but not moderate.

Higher categories can be k-light or k-moderate, for $k \in \mathbb{N}$.

Example: Yoneda lemma

For a (light) category $\mathcal{C}$ we have an isomorphism

$$Fc \cong [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}](\mathcal{Y}c, F) \quad \text{natural in } c \text{ and } F$$

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Diagrammatically:

$$\begin{array}{ccc}
\mathcal{C}^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] & \xrightarrow{\mathcal{Y} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]} & [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}]^{\mathsf{op}} \times [\mathcal{C}^{\mathsf{op}}, \mathbf{Set}] \\
{\scriptstyle \mathsf{app}} \downarrow & \cong & \downarrow {\scriptstyle \mathsf{hom}} \\
\mathbf{Set} & \hookrightarrow & \mathbf{Class}_2
\end{array}$$

Bimodules

Let $\mathcal{C}$ and $\mathcal{D}$ be (light) categories.

A bimodule $\mathcal{O}\colon \mathcal{C} \nrightarrow \mathcal{D}$ provides

- for each $c \in \mathcal{C}$ and $d \in \mathcal{D}$, a set $\mathcal{O}(c, d)$ of morphisms $g\colon c \to d$
- composite morphisms $c' \xrightarrow{f} c \xrightarrow{g} d$ and $c \xrightarrow{g} d \xrightarrow{h} d'$.

These must satisfy two identity and three associativity laws.

Questions about bimodules

1. Should we think of a bimodule $\mathcal{C} \nrightarrow \mathcal{D}$ as a functor $\mathcal{C}^{\mathsf{op}} \times \mathcal{D}$ to $\mathbf{Set}$?
2. Should we think of it as a generalized ("pro") functor?
3. Should we think of it as going from $\mathcal{D}$ to $\mathcal{C}$?
4. Should we compose bimodules?

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I prefer not to.

From a functor to a bimodule

Two ways of constructing a bimodule $\mathcal{C} \nrightarrow \mathcal{D}$.

- A functor $F : \mathcal{C} \to \mathcal{D}$ gives

$$F^{\mathsf{Left}}(c, d) \stackrel{\mathrm{def}}{=} \mathcal{D}(Fc, d)$$

  Contravariant on 2-cells

- A functor $G : \mathcal{D} \to \mathcal{C}$ gives

$$G^{\mathsf{Right}}(c, d) \stackrel{\mathrm{def}}{=} \mathcal{C}(c, Gd)$$

  Contravariant on 1-cells

Left and right representations

For a bimodule $\mathcal{O}:\mathcal{C} \nrightarrow \mathcal{D}$,

- a *left representation* consists of a functor $F:\mathcal{C} \to \mathcal{D}$ and isomorphism $m : \mathcal{O} \cong F^{\mathsf{Left}}$
- a *right representation* consists of a functor $G:\mathcal{C} \to \mathcal{D}$ and isomorphism $n : \mathcal{O} \cong G^{\mathsf{Right}}$.

Adjunction

An adjunction of functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \quad (F \dashv G)$$

is a bimodule isomorphism $F^{\mathsf{Left}} \cong G^{\mathsf{Right}}$.

Adjunction

An adjunction of functors

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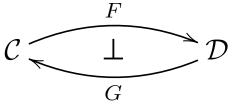

is a bimodule isomorphism $F^{\mathsf{Left}} \cong G^{\mathsf{Right}}$.

An adjunction from $\mathcal{C}$ to $\mathcal{D}$ consists of a bimodule $\mathcal{O}\colon \mathcal{C} \nrightarrow \mathcal{D}$
with a left representation (F, R)
and right representation (G, S).

One-sided modules

Let $\mathcal{C}$ be a (light) category.

A module $\mathcal{O}\colon \mathcal{C} \nrightarrow$ provides

- for each $c \in \mathcal{C}$, a set $\mathcal{O}(c)$ of morphisms $g\colon c \to$
- composite morphisms $c' \xrightarrow{f} c \xrightarrow{g} d$

These must satisfy the identity and associativity laws.
Dually for a module $\mathcal{O}\colon \nrightarrow \mathcal{C}$.

Questions about one-sided modules

1. Should we think of a module $\mathcal{C} \nrightarrow$ as a functor $\mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$?
2. Should we think of it as a bimodule $\mathcal{C} \nrightarrow 1$?
3. Or as a module $\nrightarrow \mathcal{C}^{\mathsf{op}}$?
4. Dually for a module $\nrightarrow \mathcal{C}$.

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From an object to a module

An object $v \in \mathcal{C}$ gives $v^{\mathsf{From}} \colon \nrightarrow \mathcal{C}$ and $v^{\mathsf{To}} \colon \mathcal{C} \nrightarrow$.

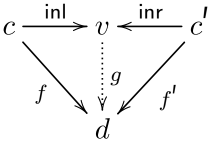
$$\begin{aligned} v^{\mathsf{From}}(c) &\stackrel{\mathrm{def}}{=} \mathcal{C}(v, c) \\ v^{\mathsf{To}}(c) &\stackrel{\mathrm{def}}{=} \mathcal{C}(c, v) \end{aligned}$$

- A representation for $\mathcal{O} \colon \nrightarrow \mathcal{C}$ consists of an object $v \in \mathcal{C}$ and isomorphism $v^{\mathsf{From}} \cong \mathcal{O}$.
- Dually for $\mathcal{O} \colon \mathcal{C} \nrightarrow$.

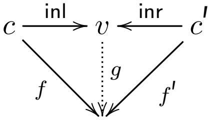
Colimits extending

A coproduct $c \xrightarrow{\mathsf{inl}} v \xleftarrow{\mathsf{inr}} c'$ in $\mathcal{C}$ is said to

- extend across $\mathcal{O} \colon \mathcal{C} \nrightarrow \mathcal{D}$ when



- extend across $\mathcal{O} \colon \mathcal{C} \nrightarrow$ when



Extension and representation

- If $\mathcal{O}: \mathcal{C} \nrightarrow$ is representable, then every colimit in $\mathcal{C}$ extends across it.
- If $\mathcal{O}: \mathcal{C} \nrightarrow \mathcal{D}$ is right representable, then every colimit in $\mathcal{C}$ extends across it.

This is half of the theorem that left adjoints preserve colimits.

General theory of modules

- So far we have dealt with ordinary categories, and modules between them.
- More generally, for a 2-moderate multicategory $\mathcal{W}$, we can speak of $\mathcal{W}$-enriched categories, and modules between them.

Background

- Semantics of call-by-push-value decomposes this into a strong adjunction between $\mathcal{C}$ and a locally indexed category $\mathcal{D}$.
- "Strong adjunction" is formulated using a module $\nrightarrow \mathcal{D}$, not a bimodule.
- Showing it's equivalent to locally indexed adjunction is complicated.
- I wanted a cleaner story.

Locally indexed category

Let $\mathcal{V}$ be a (light) category. A locally $\mathcal{V}$-indexed category $\mathcal{C}$ consists of

- a class ob $\mathcal{C}$ of objects
- for all $x \in \mathcal{V}$ and $c, c' \in \mathcal{C}$, a set $\mathcal{C}_x(c, c')$ of morphisms $c \xrightarrow[x]{f} c'$
- reindexing of $c \xrightarrow[x]{f} c'$ by $y \xrightarrow{k} x$ giving $c \xrightarrow[y]{k^* f} c'$
- identities $c \xrightarrow[x]{\mathsf{id}_{x,c}} c$
- composition of $c \xrightarrow[x]{f} c' \xrightarrow[x]{g} c''$ giving $c \xrightarrow[x]{f;g} c''$.

Must satisfy the evident seven equations.
Also locally $\mathcal{V}$-indexed functor, natural transformation, bimodule and module.

Alternative view: indexed

A locally $\mathcal{V}$-indexed category $\mathcal{C}$ is

- a class ob $\mathcal{C}$
- a $\mathcal{V}$-indexed category
  - whose fibres have object class ob $\mathcal{C}$
  - and whose reindexing functors are identity-on-objects.

Locally graded category (Wood)

Let $\mathcal{V}$ be a (light) category. A locally $\mathcal{V}$-graded category $\mathcal{C}$ consists of

- a class ob $\mathcal{C}$ of objects
- for all $x \in \mathcal{V}$ and $c, c' \in \mathcal{C}$, a set $\mathcal{C}_x(c, c')$ of morphisms $c \xrightarrow[x]{f} c'$
- reindexing of $c \xrightarrow[x]{f} c'$ by $y \xrightarrow{k} x$ giving $c \xrightarrow[y]{k^* f} c'$
- identities $c \xrightarrow[1]{\mathrm{id}_c} c$
- composition of $c \xrightarrow[x]{f} c' \xrightarrow[y]{g} c''$ giving $c \xrightarrow[x \otimes y]{f;g} c''$.

Must satisfy the evident seven equations.
Also locally $\mathcal{V}$-graded functor, natural transformation, bimodule and module.

Alternative view 2: enriched

For a category $\mathcal{V}$,

- we have a 2-moderate cartesian category $[\mathcal{V}^{\mathsf{op}}, \mathbf{Set}]$
- locally $\mathcal{V}$-indexed means $[\mathcal{V}^{\mathsf{op}}, \mathbf{Set}]$-enriched.

For a monoidal category $\mathcal{V}$,

- we have a 2-moderate multicategory $[\mathcal{V}^{\mathsf{op}}, \mathbf{Set}]$
- locally $\mathcal{V}$-graded means $[\mathcal{V}^{\mathsf{op}}, \mathbf{Set}]$-enriched.

Locally indexed vs locally graded

We have defined

- locally $\mathcal{V}$-indexed, for a category $\mathcal{V}$
- locally $\mathcal{V}$-graded, for a monoidal category $\mathcal{V}$.

Theorem

For cartesian $\mathcal{V}$, the two notions are equivalent.

Distributivity

Let $\mathcal{V}$ be a category with coproducts
or a monoidal category with distributive coproducts.

Let $\mathcal{C}$ be a locally $\mathcal{V}$-indexed category or locally $\mathcal{V}$-graded category.

$\mathcal{C}$ is distributive when

for $c \xrightarrow[x]{f} c'$ and $c \xrightarrow[y]{f'} c'$

there's a unique mediating map $c \xrightarrow[x+y]{f} c'$.

i.e. the coproduct extends across $\mathcal{D}_{-}(c, c')$ for all c, c'.

This corresponds to restricting the enriching multicategory $[\mathcal{V}^{\mathsf{op}}, \mathbf{Set}]$.

Universal properties in a locally $\mathcal{V}$-graded category

Coproduct	$\prod_{i \in I} \mathcal{C}_x(c_i, y)$	$\cong$	$\mathcal{C}_x(\bigoplus_{i \in I} c_i, y)$
Copower	$\mathcal{C}_{x \otimes a}(c, y)$	$\cong$	$\mathcal{C}_x(a.c, y)$
Product	$\prod_{i \in I} \mathcal{C}_x(y, c_i)$	$\cong$	$\mathcal{C}_x(y, \prod_{i \in I} c_i)$
Power	$\mathcal{C}_{x \otimes a}(y, c)$	$\cong$	$\mathcal{C}_x(y, c^a)$
Internal hom	$\mathcal{C}_x(c, d)$	$\cong$	$\mathcal{V}(x, c \multimap d)$

Locally $\mathcal{V}$-graded category: three examples (Wood)

A $\mathcal{V}$-actegory is a category $\mathcal{C}$ with monoidal action $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$.

$$\mathcal{C}^{\mathsf{Act}}_x(c, d) \stackrel{\text{def}}{=} \mathcal{C}(x \oslash c, d)$$

A $\mathcal{V}$-opactegory is a category $\mathcal{C}$ with monoidal action $\ni\!\!- : \mathcal{C} \times \mathcal{V}^{\mathsf{op}} \to \mathcal{C}$.

$$\mathcal{C}^{\mathsf{Opact}}_x(c, d) \stackrel{\text{def}}{=} \mathcal{C}(c, d \ni\!\!- x)$$

A $\mathcal{V}$-enriched category $\mathcal{C}$.

$$\mathcal{C}^{\mathsf{Enr}}_x(c, d) \stackrel{\text{def}}{=} \mathcal{V}(x, \mathcal{C}(c, d))$$

Characterizing these constructions

- $\mathcal{V}$-actegory = locally $\mathcal{V}$-graded category with copowers.
- $\mathcal{V}$-opactegory = locally $\mathcal{V}$-graded category with powers.
- $\mathcal{V}$-enriched category = locally $\mathcal{V}$-graded category with internal homs.

Maps to a locally $\mathcal{V}$-graded category $\mathcal{D}$ (Wood)

Map from a $\mathcal{V}$-actegory $\mathcal{C}$ to $\mathcal{D}$.

- A functor $H : \mathcal{C} \to \mathcal{D}_1$.
- A strength, consisting of morphisms $Hc \xrightarrow[x]{t_{x,c}} H(x \oslash c)$

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Map from a $\mathcal{V}$-enriched category $\mathcal{C}$ to $\mathcal{D}$.

- Function ob $\mathcal{C} \to$ ob $\mathcal{D}$.
- Morphisms $Hc \xrightarrow[\mathcal{C}(c,c')]{H_{c,c'}} Hc'$.

These correspond to locally $\mathcal{V}$-graded functors.

Modules from an actegory

A bimodule from a $\mathcal{V}$-actegory $\mathcal{C}$ to $\mathcal{D}$ consists of

- for each $c \in \mathcal{C}$ and $d \in \mathcal{D}$ a set $\mathcal{O}(c,d)$ of morphisms $g\colon c \to d$
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- composition $c \xrightarrow{g} d \xrightarrow[x]{h} d'$ giving a morphism $x \oslash c \xrightarrow{g;h} d'$

satisfying the evident equations.

Equivalences

- Bimodules from $\mathcal{C} \nrightarrow \mathcal{D}$ correspond to bimodules $\mathcal{C}^{\mathsf{Act}} \nrightarrow \mathcal{D}$ across which the copowers extend.
- Bimodules $\mathcal{V} \nrightarrow \mathcal{D}$ are precisely modules $\nrightarrow \mathcal{D}$.

Adjunctions

Abstractly, a model of call-by-push-value is

- a cartesian category $\mathcal{V}$ with countable distributive coproducts
- a countably distributive locally $\mathcal{V}$-graded category $\mathcal{D}$ with countable products and powers
- an adjunction between $\mathcal{V}^{\mathsf{Act}}$ and $\mathcal{D}$,
  i.e. a bimodule $\mathcal{O}$ with left and right representations.

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3. The copowers in $\mathcal{V}^{\mathsf{Act}}$ extend across $\mathcal{O}$,
   because $\mathcal{O}$ is right representable.
   So $\mathcal{O}$ corresponds to a module $\mathcal{V} \nrightarrow \mathcal{D}$
   which is precisely a module $\nrightarrow \mathcal{D}$.