# Morphisms between plays

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## In a small category $\mathbb A$

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Examples:

- Innocence
- ② Single-threadedness
- Asynchronous games

The first two examples use alternating plays.

A play *s* consists of

- an initial segment moves  $s\subseteq\mathbb{N}$
- (justification pointers) a map moves  $s \longrightarrow \{*\} + \text{moves } s$
- (arena elements) a map moves  $s \longrightarrow R$  (arena element).

with the usual conditions.

A strategy  $\sigma$  (Opponent first) on  ${\it R}$  is a set of even-length plays that

- is lower wrt the prefix order
- contains the empty play
- is deterministic.

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## Traces and divergences

- Finite trace is  $s \in \sigma$
- Divergence is  $sm \perp$  where  $s \in \sigma$  but  $\not\exists n. smn \in \sigma$ .
- Infinite trace is infinite play whose even-length prefixes are all in  $\sigma$ .

- A P-visible play is one in which Proponent always points to a move in the current P-view.
- A P-visible strategy is one in which all plays are P-visible.
- An innocent strategy is a special kind of P-visible strategy.
- It can be represented as a set of P-views. (After the first move, Opponent always points to the previous move.)

Innocent strategies are used to model state-free terms.

- A thread-visible play is one in which Proponent doesn't change thread.
- A thread-visible strategy is one in which all plays are thread-visible.
- A single-threaded strategy is a special kind of thread-visible strategy.
- It can be represented as a set of well-opened plays. (After the first move, Opponent never again points to \*).

Single-threaded strategies are used to model values.

## Traditional innocence

Doesn't work for certain language features:

- Nondeterminism, under linear time semantics
- Name generation (*v*-calculus)
- Polymorphism

In each case there are state-free terms whose denotation isn't "innocent".

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## Traditional single-threadedness

Doesn't work for polymorphism.

There are (state-free) values whose denotation isn't "single-threaded".

We form a category of P-visible plays and P-viewing morphisms.

# P-viewing morphism $s \xrightarrow{f} t$

- A function moves  $s \longrightarrow$  moves t.
- Preserves arena elements and justification pointers.
- If  $n^O$  is followed by  $m^P$  then f(n) is followed by f(m).
- If  $n^O$  is followed by  $\perp$  then f(n) is followed by  $\perp$ .

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There's an analogous theorem for OP-visible strategies.

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Conjecture: these are precisely the strategies definable by a state-free term.

We form a category of thread-visible plays and threading morphisms.

# Threading morphism $s \xrightarrow{f} t$

- A function moves  $s \longrightarrow$  moves t.
- Preserves arena elements and justification pointers.
- If  $n^O$  is followed by  $m^P$  then f(n) is followed by f(m).
- If  $n^O$  is followed by  $\perp$  then f(n) is followed by  $\perp$ .
- If m<sup>P</sup> is thread-followed by n<sup>O</sup> (there might be moves in between, but not from the same thread) then f(m) is thread-followed by f(n).

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both its finite trace set and its divergence set are lower wrt threading morphisms.

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Laird, Ghica, Murawski gave a game semantics for concurrent calculi, up to may-testing, as follows.

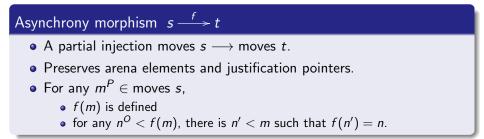
A strategy  $\sigma$  on R is a set of non-alternating plays

- lower wrt the prefix order
- containing the empty play
- (P-swap) if  $sm^P nt \in \sigma$  then  $snm^P t \in \sigma$  (assuming it's a play)
- (O-swap)if  $smn^{O}t \in \sigma$  then  $sm^{O}nt \in \sigma$  (assuming it's a play)
- (O-completeness) if  $s \in \sigma$  then  $sm^O \in \sigma$ .

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- Challenging to adapt to infinite plays, where we might want to perform infinitely many swaps simultaneously

## We form a category of non-alternating plays and asynchrony morphisms.



A set of non-alternating plays is an asynchronous strategy iff

it is lower wrt asynchrony morphisms and contains the empty play.

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it is lower wrt asynchrony morphisms and contains the empty play.

A similar result was proved by Mohamed Menaa, and used to establish the categorical properties of the asynchronous game model.