The Price of Mathematical Scepticism

Paul Blain Levy

University of Birmingham

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Outline

My thesis

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- The price of reality scepticism is consistency scepticism.

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- But our beliefs about reality, bivalence, choice and consistency should all be aligned.
- Believing in the consistency of everything and the reality of nothing is not an option.
- The price of reality scepticism is consistency scepticism.
- Qualification: in some settings, consistency can be established by other means.

Theories

$$\mathrm{PA} \subseteq \mathrm{Z}_2 \subseteq \mathrm{Z}_3 \subseteq \mathrm{ZF}$$

- Peano Arithmetic (PA) is a theory of natural numbers $0, 1, 2, \ldots$
- Second-order arithmetic (Z_2) is a theory of $\mathbb{N}$ and $\mathcal{P}\mathbb{N}$.
- Third-order arithmetic (Z_3) is a theory of $\mathbb{N}$ and $\mathcal{P}\mathbb{N}$ and $\mathcal{P}\mathcal{P}\mathbb{N}$.
- Zermelo-Fraenkel (ZF) is a general theory of sets.

Two kinds of choice

A relation R from A to B is entire when every $x \in A$ has an R-image.

Axiom of Choice (AC)

For any entire relation R from A to B,
there's a function $f : A \to B$ such that $\forall x \in A.\, x\, R\, f(x)$.

Dependent Choice (DC)

For any entire relation R from A to A, and any $a \in A$,
there's a sequence $(x_n)_{n \in \mathbb{N}}$ in A such that $x_0 = a$ and $\forall n \in \mathbb{N}.\, x_n\, R\, x_{n+1}$.

AC implies DC (given ZF).

Banach-Tarski theorem

- Banach-Tarski: One unit ball can be transformed into two unit balls, by partitioning it into five subsets and rigidly moving each of them.
- This is provable in ZF+AC.
- Not provable in ZF+DC assuming ZFC with an inaccessible is consistent.

Continuum Hypothesis

- We write $2^{\mathbb{N}}$ for the set of all bitstreams, e.g. 100111...
- The Continuum Hypothesis (CH) says that every uncountable set of bitstreams is equinumerous with $2^{\mathbb{N}}$.
- (Gödel, Cohen, Lévy, Solovay) It's impossible to prove or refute CH in ZFC provided ZFC is consistent, and large cardinal hypotheses don't help.
- So it seems to be unknowable whether CH is true.
- Although some people (e.g. Woodin) have advocated principles that imply CH or imply $\neg$ CH, these are controversial and beyond the scope of this talk.

Classical beliefs about CH and AC

- The Continuum Hypothesis (CH) is bivalent—either objectively true or objectively false. Even if it is absolutely unknowable which of these is the case.
- AC is true.

Some alternative views

- "There is no canonical universe of mathematical reality, but rather many universes of equal status. All of them satisfy the ZFC axioms, but CH holds in some of them and fails in others."
- "AC is unacceptable because it leads to the Banach–Tarski theorem. Therefore ZF+DC should be adopted as a foundational theory."

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Each favours a strong foundational theory (at least ZF), yet at the same time is sceptical of the classical conception.

My thesis: we cannot "have our cake and eat it" in this way.

Third-order arithmetic

CH and Banach-Tarski are third-order arithmetical statements: all quantifiers range over $\mathbb{N}$ or $\mathcal{P}\mathbb{N}$ or $\mathcal{P}\mathcal{P}\mathbb{N}$.

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To discuss them, we need not consider an advanced theory such as ZF. Just Z_3, the theory of third-order arithmetic.

My thesis: the headline

Insofar as we doubt the bivalence of CH or the truth of AC, we should also doubt the consistency of Z_3.

Likewise, doubting DC leads to doubt in the consistency of Z_2.

Models without AC

In some fields of mathematics, such as topos theory, it is common to avoid AC and other classical principles, in order to gain information about interesting models where these principles fail.

People doing this may still believe AC to be true in reality, just not in their models of interest.

According to the view that I'm presenting, that's fine.

Principles of justified belief

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So my claim that certain people ought to doubt the consistency of Z_3
doesn't mean that they should believe Z_3 to be inconsistent.
They should not.

Doubt is our default position

The Cleopatra hypothesis

Over the course of her life, Cleopatra ate an even number of grapes.

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No evidence for or against.

Although a person who believes this may happen to be right,
such belief is arbitrary and unjustified.

The correct position is to doubt it.

Example: Goldbach and variations

Goldbach conjecture

"Every even natural number other than 0 and 2 is a sum of two primes."

This has been checked for numbers $\leqslant 4.01 \times 10^{18} + 2$. (Oliveira e Silva, 2022)

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Goldbach variants

- Googolplex Goldbach: Every even Googolplex-bounded number other than 0 and 2 is a sum of two primes.
- Liminal Goldbach: The least even number that hasn't yet been checked is a sum of two primes.

Proof + intuition

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What would cause us to believe it? Either a proof, or intuition, or a combination of the two. These are (we shall suppose) the only acceptable grounds for belief.

Furthermore, appeals to intuition raise the tricky question of which intuitions are reliable.

Inductive evidence: reject!

We might be tempted towards belief by the weight of inductive evidence.

But even Liminal Goldbach might be false for all we know. So we doubt it.

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But even Liminal Goldbach might be false for all we know. So we doubt it.

Inductive evidence, however strong, is not adequate grounds for belief.
Mathematicians throughout the ages have largely agreed on this point.

Inductive evidence: accept?

Inductive evidence is often taken seriously in mathematics. For example:

- In number theory, to support the Goldbach conjecture.
- In computational complexity theory, to support the $P \neq NP$ hypothesis.
  "Invisible fence."
- In set theory, to support the $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ hypothesis.
  "Extrinsic justification."

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Henceforth, we follow the traditional view: inductive evidence is not adequate grounds for belief.

Consistency statements

Let $\mathcal{T}$ be a theory.

- The assertion $\text{Con}(\mathcal{T})$ says that $\mathcal{T}$ is consistent, i.e. False is unprovable in T.
- The assertion $\text{Con}_{\mathsf{G}}(\mathcal{T})$ says that $\mathcal{T}$ is Googolplex-consistent, i.e., False has no proof whose length is Googolplex-bounded.

I assume that proof length has been precisely defined for each of our theories.

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As before, our default position is to doubt it, and only proof or intuition will give us adequate grounds to believe it.

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Take for example $\mathsf{Con}_{\mathsf{G}}(\mathrm{Z}_3)$.

As before, our default position is to doubt it, and only proof or intuition will give us adequate grounds to believe it.

Gödel's second incompleteness theorem and similar results do not justify relaxing this policy.

The Clever People argument

One sometimes hears the following argument for consistency.

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(Hamkins) The negation of Fermat's Last Theorem turned out to be inconsistent, even though, before Wiles, many clever people had looked seriously and been unable to refute it.

Summary

- For any statement, our default position is doubt.
- Only proof and/or intuition will move us to a state of belief.
- We need to decide which intuitions are reliable.
- Inductive inference is not accepted.
- These principles apply, in particular, to consistency statements.

A list of open problems

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Physical sentences

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Computational sentences

Googolplex Goldbach Every even Googolplex-bounded number other than 0 and 2 is a sum of two primes.

$$\forall n \leqslant 10^{10^{100}}. \phi(n)$$

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$$\forall n \leqslant 10^{10^{100}}.\, \phi(n)$$

All the other examples come in pairs:

- a single-quantifier sentence ($\forall$ or $\exists$)
- and a double-quantifier sentence ($\forall\exists$ or $\exists\forall$).

Arithmetical sentences

Quantifiers range over $\mathbb{N}$.

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Goldbach conjecture

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Such a sentence is Π^0_1 or falsifiable.

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Twin prime conjecture

There are infinitely many pairs $(n, n+2)$ of prime numbers.

$$\forall n \in \mathbb{N}.\, \exists m \in \mathbb{N}.\, \phi(n, m) \qquad \phi \text{ is computational.}$$

Such a sentence is Π^0_2.

Quantifiers range over $2^{\mathbb{N}}$.

Second-order arithmetical sentences

Quantifiers range over $2^{\mathbb{N}}$.

Littlewood conjecture

For any real numbers α and β, we have $\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0$, where $\|\ \|$ is the distance to the nearest integer.

$$\forall x \in 2^{\mathbb{N}}.\, \phi(x) \qquad \phi \text{ is arithmetical.}$$

Such a sentence is Π^1_1.

Toeplitz conjecture

Every simple closed curve contains all four vertices of some square.

$$\forall x \in 2^{\mathbb{N}}.\, \exists y \in 2^{\mathbb{N}}. \phi(x, y) \qquad \phi \text{ is arithmetical.}$$

Such a sentence is Π^1_2.

Third-order arithmetical sentences

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Continuum Hypothesis

There's a bijection from $\aleph_1$ to $2^{\mathbb{N}}$.

$$\exists x : 2^{\mathbb{N}} \to 2.\, \phi(x) \qquad \phi \text{ is second-order arithmetical.}$$

Such a sentence is Σ^2_1.

Suslin Hypothesis

The tree $\{0,1\}^{<\omega_1}$ has no subtree in which every chain and every antichain is countable.

$$\forall x : 2^{\mathbb{N}} \to 2. \exists y : 2^{\mathbb{N}} \to 2.\, \phi(x,y) \qquad \phi \text{ is second-order arithmetical.}$$

Such a sentence is Π^2_2.

Unrestricted set-theoretic sentences

Quantifiers range over Ord, the class of all ordinals.

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Generalised Continuum Hypothesis (GCH)

Every infinite cardinal κ satisfies $\kappa^+ = 2^\kappa$.

$\forall \kappa \in \mathsf{Ord}.\, \phi(\alpha)$ ϕ is restricted.

"Restricted" means that each quantifier ranges over a set.
The $\mathcal{P}$ construction may be used.

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Eventually Generalised Continuum Hypothesis

There's an infinite cardinal λ such that every cardinal $\kappa \geqslant \lambda$ satisfies $\kappa^+ = 2^\kappa$.

$\exists \lambda \in \mathsf{Ord}.\, \forall \kappa \in \mathsf{Ord}.\, \phi(\lambda, \kappa)$ $\quad$ ϕ is restricted.

Class-theoretic sentences

Quantifiers range over 2^{Ord}, the collection of all long bitstreams.

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Club-Failure Hypothesis (Schlutzenberg)

Every club class of infinite cardinals has a member whose successor cardinal κ is a GCH failure, i.e. satisfies $\kappa^+ < 2^\kappa$.

$$\forall X \in 2^{\mathsf{Ord}}.\, \phi(X) \qquad \phi \text{ is set-theoretic.}$$

Ord-Suslin Hypothesis (Hamkins and Switzer)

The tree $\{0,1\}^{<\mathsf{Ord}}$ has no subtree in which every chain and every antichain is a set.

$$\forall X \in 2^{\mathsf{Ord}}.\, \exists Y \in 2^{\mathsf{Ord}}.\, \phi(X, Y) \qquad \phi \text{ is set-theoretic.}$$

Optimistic scenario

What are the prospects for solving these problems?

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Here is an optimistic scenario.

- New archaeological techniques will resolve the Cleopatra hypothesis. ("Resolve" means "prove or refute".)
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Next, a pessimistic scenario.

Pessimistic scenario

- The Cleopatra hypothesis can't be resolved in any way.
- Googolplex Goldbach can't be resolved in any way within the lifetime of the universe.
- Goldbach can't be reduced in any way to a computational sentence within the lifetime of the universe, and can't be proved in any way.

And for the rest:

- Twin Prime, which is $\forall\exists$, can't be reduced in any way to a $\exists\forall$ sentence.
- Littlewood can't be reduced in any way to an arithmetical sentence.
- And so forth.

Rough summary: All of our sentences are absolutely unknowable.

The bivalence questionnaire

Now I am going to interrogate you.

Assume the pessimistic scenario, or at least bear in mind that it's possible.

Which of our sentences do you think are bivalent?

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Which of our sentences do you think are bivalent?

In other words, do you think that—despite our hopeless ignorance—there is a fact of the matter whether Cleopatra ate an even number of grapes?

Whether every even Googolplex-bounded number other than 0 and 2 is a sum of two primes?

And so forth.

Guidance notes

Some things to bear in mind when answering the questions:

- Answering Yes means you think the sentence is bivalent.
- Answering No means you doubt the bivalence—an absence of belief. It doesn't mean you positively think the sentence isn't bivalent.

Guidance notes

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Bivalence ambivalence is allowed, and even encouraged.

A measuring device

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They provide a crude but useful device to measure a person's belief in objective reality, a belief known as "realism" or "platonism". (These words will be used interchangeably.)

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- Particularist: Third-order arithmetical sentences and higher are bivalent, AC is true.
- Totalist: Unrestricted sentences are bivalent.
- Views that accept the bivalence of class-theoretic sentences are beyond the scope of this talk.

Questions

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- Is this all just a choice between various coloured pills?
- Why not have an option for someone who accepts bivalence of 17th order but not 18th order?
- Or for someone who accepts the bivalence of Π^0_{52} but not Π^0_{53}?

Answers

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- Belief should not be arbitrary.
- Furthermore (according to our principles), it should justified by proof or intuition.
- So we cannot be mere truth value realists, believing for no reason that certain sentences are bivalent.
- What, then, are the intuitions that support the various positions?

Intuitions of mathematical reality

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They are little people inside our head, and each of them is going to speak.

For the moment, just listen to them.

We postpone the question of whether they are reliable.

The intuitions

Googolplex

"I perceive the notion of Googolplex-bounded number. Since this is a clearly defined notion, quantification over the set $\mathbb{N}_G$ yields an objective truth value."

Arbitrary Natural Number

"I perceive the notion of a natural number, given by zero and successor. This is a clearly defined notion, as restrictive as possible. So quantification over the set $\mathbb{N}$ yields an objective truth value."

The intuitions (continued)

Arbitrary Sequence

"Given a set B, I perceive the notion of a sequence $(x_n)_{n\in\mathbb{N}}$ in B, which consists of successive arbitrary choices of an element of B. This is a clearly defined notion, as liberal as possible. So quantification over the set $B^{\mathbb{N}}$ yields an objective truth value. Since a sequence consists of successive arbitrary choices, DC holds."

Arbitrary Function

"Given sets A and B, I perceive the notion of a function $f : A \to B$, which consists of independent arbitrary choices $f(a) \in B$, one for each $a \in A$. This is a clearly defined notion, as liberal as possible. So quantification over the set B^A yields an objective truth value. Since a function consists of independent arbitrary choices, AC holds."

The intuitions (continued)

Arbitrary Ordinal

"I perceive the notion of an ordinal. This is a clearly defined notion, as liberal as possible. So quantification over the class Ord yields an objective truth value."

Schools and intuitions

Each school draws a line between the credible and doubtful intuitions.

School	Accepts
Ultrafinitism	Nothing
Finitism	Googolplex
Countabilism	Arbitrary Natural Number
Sequentialism	Arbitrary Sequence
Particularism	Arbitrary Function
Totalism	Arbitrary Ordinal

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This taxonomy is crude and ignores finer distinctions, e.g. between finitists and constructivists/intuitionists.

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My line of thinking does not allow this. For if Arbitrary Natural Number is an unreliable intuition, then even the bivalence of the Goldbach conjecture is in doubt.

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Likewise for each pair of sentences in our questionnaire: we either answer Yes to both or No to both.

The attack on platonism

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In response to this attack, I will make the following points.

1. Platonism is not essentialism.
2. Rejecting platonism leads to ultrafinitism.

Essentialism

What does it mean to "believe in the reality of X", where X is a totality such as $\mathbb{N}_{\mathsf{G}}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathcal{P}\mathcal{P}\mathbb{N}$ or Ord?

(I mention $\mathbb{C}$ because of the interchangeable nature of i and $-i$.)

Essentialism

What does it mean to "believe in the reality of X", where X is a totality such as $\mathbb{N}_{\mathsf{G}}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathcal{P}\mathcal{P}\mathbb{N}$ or Ord?

(I mention $\mathbb{C}$ because of the interchangeable nature of i and $-i$.)

Must we believe that each natural number, complex number, ordinal etc. exists "out there" with a transcendent essence?

Call this belief essentialism.

Platonism is not essentialism

Define two simple encodings $\mathbb{Q} \to \mathbb{N}$, dubbed red and yellow. The rational $\frac{1}{2}$ has red encoding 18 and yellow encoding 90.

Via the red encoding, and also via the yellow encoding, any believer in the reality of $\mathbb{N}$ must also believe in that of $\mathbb{Q}$.

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Thus, while people often say "I believe in the reality of $\mathbb{N}$ but have doubts about $\mathbb{R}$," nobody says "I believe in the reality of $\mathbb{N}$ but have doubts about $\mathbb{Q}$."

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Summary: be careful when construing platonism. Mere truth value realism is too little, but essentialism is too much.

Rejecting platonism necessarily leads to ultrafinitism

If we still think platonism is flawed (e.g. because of the access problem), then we have to doubt all the intuitions and hence the bivalence of Googolplex Goldbach. The ultrafinitist view.

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This is contentious, because finitists and constructivists sometimes argue in just this way against more credulous positions.

But the anti-platonist argument has nothing to do with infinity *per se*. The set $\mathbb{N}_{\mathsf{G}}$ is no more capable of direct apprehension, by an actual human or computer, than $\mathbb{N}$.

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Welcome to the club!

How far to go?

Leaving aside ultrafinitists, then, we all accept $\mathbb{N}_{\mathsf{G}}$ and are platonists. But how far shall we go? The intuitions are profoundly different.

- The entire set $\mathbb{N}_{\mathsf{G}}$ can in principle be grasped.
- Each natural number can in principle be grasped.
- A sequence is given by just one choice at a time.
- A function from A is given by A-many choices at the same time.

Arbitrary Function: the debate

Suppose we accept Arbitrary Natural Number and Arbitrary Sequence. Shall we accept Arbitrary Function? Two reasons are sometimes given for not doing so.

1. Independence is a reason to doubt CH bivalence.
2. Banach-Tarski is a reason to doubt AC.

I will argue against these reasons.

The independence issue

Even if the truth value of CH is absolutely unknowable, this is no argument against bivalence. For analogy, whether Cleopatra ate an even number of grapes is unknowable, but I don't consider this an argument against bivalence.

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In fact, none of the intuitions claim to have complete knowledge of the entities they perceive. On the contrary, they merely claim to know the most basic properties.

Most of our knowledge about $\mathbb{N}$, for example, comes from proof, not directly from the Arbitrary Natural Number intuition. Whatever limits may exist on our proof ability, they do not call into question the reliability of that intuition.

The Banach-Tarski issue

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Furthermore, it has been argued that there are also theorems provable without AC that violate geometric intuition.

My view on Arbitrary Function

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Personally I find the intuition strong enough to accept, but am not free of ambivalence, and can understand others being more cautious.

The problem with Arbitrary Ordinal

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Personally, I am sceptical.

Totalists accept it, but have a similar problem with quantification ranging over 2^{Ord}.

Summary so far

- The starting position was that the only acceptable grounds for belief are proof and intuition.
- I listed some intuitions that I experience (and am assuming that there are no others that would undermine my argument).
- The key question was which of these are reliable, and noted various possible answers.
- Now let's consider their consequences.

Arbitrary Natural Number: acceptance

If we accept Arbitrary Natural Number, we believe in a platonic realm of natural numbers, and the bivalence of every arithmetical statement.

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Provided we can reflect on our reasoning, we see that we accept every PA theorem, and accept $\text{Con}(\mathrm{PA})$.

Arbitrary Natural Number: doubt

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Unless we accept one of these principles, we have to doubt Con(PA), and indeed Con$_\mathsf{G}(\mathrm{PA})$.

Consistency of Z_2

If we accept (Arbitrary Natural Number and) Arbitrary Sequence, and can reflect on our reasoning, then we believe Con(Z_2).

But if we doubt Arbitrary Sequence, i.e. are countabilists, then the simple path to consistency is blocked. Perhaps some other proof will convince us.

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Unless we accept this principle—which is rather unlikely—we have to doubt $\mathsf{Con}_{\mathsf{G}}(\mathrm{Z}_2)$.

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So we have to doubt $\mathsf{Con}_{\mathsf{G}}(\mathrm{Z}_3)$. There is no middle ground.

All or nothing

My key point is that, although we can either accept or doubt an intuition, we cannot half-accept. If we consider an intuition to be unreliable, then we should fully discard it.

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Historical example

Once the mathematical community came to view geometrical intuition as unreliable, it was fully discarded, in the sense that appealing to it in a proof was no longer allowed.

Doubting Choice

Accepting Z_3 but not AC is not an option, since AC is asserted by Arbitrary Function.

So if the Banach-Tarski theorem is anything less than an objectively true statement, then either Arbitrary Natural Number or Arbitrary Function is an unreliable intuition, and $\mathsf{Con}_{\mathsf{G}}(\mathrm{Z}_3)$ is in doubt.

In the same way, DC is asserted by Arbitrary Sequence, so doubt in DC leads to doubt in $\mathsf{Con}_{\mathsf{G}}(\mathrm{Z}_2)$.

Finitist vs ultrafinitist

Primitive Recursion Arithmetic (PRA)

A "finitistically acceptable" fragment of Peano arithmetic. Skolem, Goodstein, Tait)

Every quantifier must be bounded by a (computed) natural number.

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Every quantifier must be bounded by a (computed) natural number.

A finitist who can reflect on their reasoning will accept Con(PRA).

By contrast, an ultrafinitist has to doubt even $\mathsf{Con}_{\mathsf{G}}(\mathrm{PRA})$.

Historical examples

- Silver doubted Con(Z_3).
- Gentzen, Lorenzen and Péter doubted Con(Z_2).
- The finitist Goodstein doubted Con(PA).
- The ultrafinitist Nelson doubted Con(PRA).

Intuitionistic theories

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- Heyting arithmetic is PA without Excluded Middle. It's equiconsistent with PA.
- Intensional Intuitionistic Second-Order Arithmetic is Z_2 without Excluded Middle and Extensionality. It's equiconsistent with Z_2.
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Intuitionstic logic and intensionality don't help to achieve consistency.

The idea of a multiverse

Multiversism is a particular kind of bivalence scepticism.
(Skolem, Mostowski, Putnam, Hamkins and others.)

It asserts that there are many mathematical universes, all of equal status.
In short, "reality is indeterminate".

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Sometimes an analogy is drawn with the Parallel Postulate.

My criticism

I make two criticisms of the multiverse idea.

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Caveat

My criticisms apply to multiversism as a philosophical view of reality, not as a mathematical account of a class of models.

The consistency issue

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As I have argued, one who doubts the bivalence of CH ought to doubt the consistency of ZFC.

Scepticism is not multiversism

We have seen many kinds of bivalence scepticism.

- A finitist doubts the bivalence of arithmetical sentences.
- A countabilist doubts the bivalence of second-order arithmetical sentences.
- A sequentialist doubts the bivalence of third-order arithmetical sentences.
- A particularist douts the bivalence of unrestricted sentences.
- A totalist doubts the bivalence of class-theoretic sentences.

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None of these people thinks of reality as indeterminate.

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This stems from a fear that $\mathbb{N}$ may be unreal, not from a fear that it may be indeterminate.

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For, in their view, if the conjecture holds in some "version of $\mathbb{N}$" that's at least a model of PA, then it is simply true.

Countabilist doubt is not multiversism

A countabilist doubts the bivalence of the Littlewood conjecture.

$\forall x \in 2^{\mathbb{N}}.\, \phi(x)$ $\qquad$ ϕ is arithmetical.

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This stems from a fear that $2^{\mathbb{N}}$ may be unreal, not from a fear that it may be indeterminate.

For, in their view, if the conjecture fails in some "version of $2^{\mathbb{N}}$" that is at least a set of bitstreams, then it is simply false

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This stems from a fear that $2^{2^{\mathbb{N}}}$ may be unreal, not from a fear that it may be indeterminate.

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This stems from a fear that Ord may be unreal, not from a fear that it may be indeterminate.

For, in their view, if GCH fails in some "version of Ord" that is at least a class of ordinals, then it is simply false.

Totalist doubt is not multiversism

A totalist doubts the bivalence of the Club-Failure hypothesis.

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Totalist doubt is not multiversism

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This stems from a fear that 2^{Ord} may be unreal, not from a fear that it may be indeterminate.

For, in their view, if the hypothesis holds in some "version of 2^{Ord}" that is at least a collection of long bitstreams, then it is simply false.

Just beyond the boundary

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As we saw in each case, doubt in the bivalence of such a sentence cannot be ascribed to a fear that reality may be indeterminate.

Model theory vs reality

Model theory is, of course, an important field of study.

In the broad sense, it includes

- models of arithmetic
- models of set theory
- categorical models such as toposes.

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Multiversism originated from seeing model theory as a philosophical view of reality.

Caution is needed here. For mathematics is no more the study of models than astronomy is the study of telescopes.

Future work

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What should particularists and totalists doubt?

Mahlo's principle: The class of all regular ordinals is stationary.

Suggestion

Particularists (like me) should doubt $\mathsf{Con}_{\mathsf{G}}(\mathrm{ZF} + \mathrm{Mahlo})$.

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Totalists should doubt $\mathsf{Con_G}(\mathrm{KM} + \mathrm{Mahlo})$.

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Totalists should doubt $\text{Con}_\text{G}(\mathrm{KM} + \mathrm{Mahlo})$.

Suggestion

Both schools should doubt Analytic Determinacy.

Conclusions

- In thinking about mathematics, we have some freedom to decide how credulous or sceptical to be, based on the strength of our intuition and our degree of caution.
- But our beliefs about reality, bivalence, choice and consistency should all be aligned.
- Believing in the consistency of everything and the reality of nothing is not an option.

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- But our beliefs about reality, bivalence, choice and consistency should all be aligned.
- Believing in the consistency of everything and the reality of nothing is not an option.

Scepticism always comes at a price.