Similarity Quotients as Final Coalgebras

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Abstract. We give a general framework connecing a branching time relation on nodes of a transition system to a final coalgebra for a suitable endofunctor. Examples of relations treated by our theory include bisimilarity, similarity, upper and lower similarity for transition systems with divergence, similarity for discrete probabilistic systems, and nested similarity. Our results describe firstly how to characterize the relation in terms of a given final coalgebra, and secondly how to construct a final coalgebra using the relation.

Our theory uses a notion of "relator" based on earlier work of Thijs. But whereas a relator must preserve binary composition in Thijs' framework, it only laxly preserves composition in ours. It is this weaker requirement that allows nested similarity to be an example.

1 Introduction

A series of influential papers including [1,12,18,19,20] have developed a coalgebraic account of bisimulation, based on the following principles.

- A transition system may be regarded as a coalgebra for a suitable endofunctor F on **Set** (or another category).
- Bisimulation can be defined in terms of an operation on relations, called a "relational extension" or "relator".
- This operation may be obtained directly from F, if F preserves quasi-pullbacks [4].
- Given a final F-coalgebra, two nodes of transition systems are bisimilar iff they have the same *anamorphic image*–i.e. image in the final coalgebra.
- Any coalgebra can be quotiented by bisimilarity to give an *extensional* coalgebra—one in which bisimilarity is just equality.
- One may construct a final coalgebra by taking the extensional quotient of a sufficiently large coalgebra.

Thus a final F-coalgebra provides a "universe of processes" according to the viewpoint that bisimilarity is the appropriate semantic equivalence.

More recently [3,5,8,13,14,23] there have been several coalgebraic studies of simulation, in which the final F-coalgebra carries a preorder. This is valuable for someone who wants to study bisimilarity and similarity together: equality represents bisimilarity, and the preorder represents similarity. But someone who

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is exclusively interested in similarity will want the universe of processes to be a poset: if two nodes are mutually similar, they should be equal. In this paper we shall see that such a universe is also a final coalgebra, for a suitable endofunctor H on the category of posets.

For example, consider countably branching transition systems. In this case, we shall see that H maps a poset A to the set of countably generated lower sets, ordered by inclusion. A final H-coalgebra is a universe for similarity, in two senses.

- On the one hand, we can *use* a final H-coalgebra to characterize similarity, by regarding a transition system as a discretely ordered H-coalgebra.
- On the other hand, we can *construct* a final H-coalgebra, by taking a sufficiently large transition system and quotienting by similarity.

We give this theory in Sect. 4. But first, in Sect. 3, we introduce the notion of relator, which gives many notions of simulation, e.g. for transition systems with divergence and Markov chains. Finally, in Sect. 5 we look at the example of 2-nested simulation; this requires a generalization of our theory where relations are replaced by indexed families of relations.

2 Mathematical Preliminaries

Definition 1. *(Relations)*

1. *For sets X and Y, we write* $X \xrightarrow{\mathcal{R}} Y$ *when $\mathcal{R}$ is a relation from X to Y, and* $\mathrm{Rel}(X,Y)$ *for the complete lattice of relations ordered by inclusion.*
2. $X \xrightarrow{(=_X)} X$ *is the equality relation on X.*
3. *Given relations* $X \xrightarrow{\mathcal{R}} Y \xrightarrow{\mathcal{S}} Z$ *, we write* $X \xrightarrow{\mathcal{R};\mathcal{S}} Z$ *for the composite.*
4. *Given functions* $Z \xrightarrow{f} X$ *and* $W \xrightarrow{g} Y$ *, and a relation* $X \xrightarrow{\mathcal{R}} Y$ *, we write* $Z \xrightarrow{(f,g)^{-1}\mathcal{R}} W$ *for the inverse image* $\{(z,w) \in Z \times W \mid f(z)\ \mathcal{R}\ g(w)\}$.
5. *Given a relation* $X \xrightarrow{\mathcal{R}} Y$ *, we write* $Y \xrightarrow{\mathcal{R}^{\mathsf{c}}} X$ *for its converse. $\mathcal{R}$ is* difunctional *when* $\mathcal{R};\mathcal{R}^{\mathsf{c}};\mathcal{R} \subseteq \mathcal{R}$.

Definition 2. *(Preordered sets)*

1. *A* preordered set *A is a set A_0 with a preorder $\leqslant_A$. It is a* poset *(*setoid*,* discrete setoid*) when $\leqslant_A$ is a partial order (an equivalence relation, the equality relation).*
2. *We write* **Preord** *(***Poset***,* **Setoid***,* **DiscSetoid***) for the category of preordered sets (posets, setoids, discrete setoids) and monotone functions.*
3. *The functor* $\Delta : \mathbf{Set} \longrightarrow \mathbf{Preord}$ *maps X to $(X, =_X)$ and* $X \xrightarrow{f} Y$ *to f. This gives an isomorphism* $\mathbf{Set} \cong \mathbf{DiscSetoid}$.

4. *Let A and B be preordered sets. A* bimodule $A \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B$ *is a relation such that* $(\leqslant_A);\mathcal{R};(\leqslant_B) \subseteq \mathcal{R}$. *We write* $\mathrm{Bimod}(A,B)$ *for the complete lattice of bimodules, ordered by inclusion. For an arbitrary relation* $A_0 \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B_0$, *its* bimodule closure $A \overset{\overline{\mathcal{R}}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B$ *is* $(\leqslant_A);\mathcal{R};(\leqslant_B)$.

Here are some general properties of preordered sets.

Lemma 1. *(Characterization of monotonicity) Let I be a set, and let A and B be preordered sets. For any function* $A_0 \xrightarrow{f} B_0$, *the following are equivalent.*

1. $A \xrightarrow{f} B$ *is monotone.*
2. $A \xrightarrow{(f,B)^{-1}(\leqslant_B)} B$ *is a bimodule.*
3. $B \xrightarrow{(B,f)^{-1}(\leqslant_B)} A$ *is a bimodule.*

Proof. Trivial.

Lemma 2. *(Properties of posets) Let I be a set and let B be a poset.*

1. *For any preordered set A and monotone functions* $A \xrightarrow{f,g} B$, *the following conditions are equivalent.*
   - $f = g$.
   - $(\leqslant_A) \sqsubseteq (f,g)^{-1}(\leqslant_B)$ *and* $(\leqslant_A) \sqsubseteq (g,f)^{-1}(\leqslant_B)$.
2. *Let A be an preordered set. Then any embedding* $B \xrightarrow{f} A$ *is injective.*
3. *Let A be an preordered set and* $A \xrightarrow{f} B$ *an injective monotone function. Then A is a poset.*

Proof. Trivial.

Definition 3. *(Quotienting)*

1. *Let A be a preordered set. For $x \in A$, its* principal lower set $[x]_A$ *is* $\{y \in A \mid y \leqslant_A x\}$. *The* quotient poset QA *is* $\{[x]_A \mid x \in A\}$ *ordered by inclusion. (This is isomorphic to the quotient of A by the equivalence relation* $(\leqslant_A) \cap (\geqslant_A)$.*) We write* $A \xrightarrow{p_A} QA$ *for the function* $x \mapsto [x]_A$.
2. *Let A and B be preordered sets and* $A \xrightarrow{f} B$ *a monotone function. The monotone function* $QA \xrightarrow{Qf} QB$ *maps* $[x]_A \mapsto [f(x)]_B$.
3. *Let A and B be preordered sets and* $A \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} B$ *a bimodule. The bimodule* $QA \overset{Q\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} QB$ *relates* $[x]_A$ *to* $[y]_B$ *iff* $x \mathrel{\mathcal{R}} y$.

Lemma 3. *(Quotienting preserves operations on bimodules)*

1. *Let A and B be preordered sets. Then we have an isomorphism of complete lattices:*
$$\mathrm{Bimod}(A, B) \cong \mathrm{Bimod}(QA, QB)$$
mapping $\mathcal{R} \mapsto Q\mathcal{R}$*, with inverse* $\mathcal{S} \mapsto (p_A, p_B)^{-1}\mathcal{S}$.
2. *Let A be an preordered set. Then* $Q(\leqslant_A) = (\leqslant_{QA})$.
3. *Let* A, B, C *be preordered sets. For any bimodules* $A \xrightarrow{\mathcal{R}} B \xrightarrow{\mathcal{S}} C$ *we have* $Q(\mathcal{R}; \mathcal{S}) = Q\mathcal{R}; Q\mathcal{S}$.
4. *Let* A, B, C, D *be preorderd sets and let* $C \xrightarrow{f} A$ *and* $D \xrightarrow{g} B$ *be monotone. For any bimodule* $A \xrightarrow{\mathcal{R}} B$ *we have* $Q((f, g)^{-1}\mathcal{R}) = (Qf, Qg)^{-1}Q\mathcal{R}$.
5. *Let A and B be preordered sets. For any bimodule* $A \xrightarrow{\mathcal{R}} B$ *we have*
$$Q((\leqslant_B); \mathcal{R}^{\mathsf{c}}; (\leqslant_A)) = (\leqslant_{QB}); (Q\mathcal{R})^{\mathsf{c}}; (\leqslant_{QA})$$

Proof. Trivial.

We give some examples of endofunctors on **Set**.

Definition 4. 1. *For any set* X *and class* K *of cardinals, we write* $\mathcal{P}^K X$ *for the set of subsets* X *with cardinality in* K. $\mathcal{P}$ *is the endofunctor on* **Set** *mapping* X *to the set of subsets of* X *and* $X \xrightarrow{f} Y$ *to* $u \mapsto \{f(x) \mid x \in u\}$. *It has subfunctors* $\mathcal{P}^{[0,\kappa)}$ *and* $\mathcal{P}^{[1,\kappa)}$ *where* κ *is a cardinal or* ∞.
2. Maybe *is the endofunctor on* **Set** *mapping* X *to* $X + 1 = \{\mathrm{Just}\ x \mid x \in X\} \cup \{\Uparrow\}$ *and* $X \xrightarrow{f} Y$ *to* $\mathrm{Just}\ x \mapsto \mathrm{Just}\ f(x), \Uparrow \mapsto \Uparrow$.
3. *A* discrete subprobability distribution *on a set* X *is a function* $d : X \longrightarrow [0, 1]$ *such that* $\sum_{x \in X} d_x \leqslant 1$ *(so d is countably supported). For any* $U \subseteq X$ *we write* $dU \stackrel{\text{def}}{=} \sum_{x \in U} d_x$*, and we write* $d \Uparrow \stackrel{\text{def}}{=} 1 - d(X)$. *D is the endofunctor on* **Set** *mapping* X *to the set of discrete subprobability distributions on* X *and* $X \xrightarrow{f} Y$ *to* $d \mapsto (y \mapsto d(f^{-1}\{y\}))$.

Definition 5. *Let* $\mathcal{C}$ *be a category.*

1. *Let* F *be an endofunctor on* $\mathcal{C}$. *An* F-coalgebra M *is a* $\mathcal{C}$*-object* $M^{\cdot}$ *and morphism* $M^{\cdot} \xrightarrow{\zeta_M} FM^{\cdot}$. *We write* $\mathrm{Coalg}(\mathcal{C}, F)$ *for the category of* F*-coalgebras and homomorphisms.*
2. *Let* F *and* G *be endofunctors on* $\mathcal{C}$*, and* $F \xrightarrow{\alpha} G$ *a natural transformation. We write* $\mathrm{Coalg}(\mathcal{C}, \alpha) : \mathrm{Coalg}(\mathcal{C}, F) \longrightarrow \mathrm{Coalg}(\mathcal{C}, G)$ *for the functor mapping* M *to* $(M^{\cdot}, \zeta_M; \alpha_{M^{\cdot}})$ *and* $M \xrightarrow{f} N$ *to* f.

Examples of coalgebras:

- a *transition system* is a $\mathcal{P}$-coalgebra
- a *countably branching transition system* is a $\mathcal{P}^{[0,\aleph_0]}$-coalgebra
- a *transition system with divergence* is a $\mathcal{P}$Maybe-coalgebra

– a *partial Markov chain* is a D-coalgebra.

There are also easy variants for labelled systems.

Lemma 4. *[9] Let $\mathcal{C}$ be a category and $\mathcal{B}$ a reflective replete (i.e. full and isomorphism-closed) subcategory of $\mathcal{C}$.*

1. *Let $A \in \mathsf{ob}\ \mathcal{C}$. Then A is a final object of $\mathcal{C}$ iff it is a final object of $\mathcal{B}$.*
2. *Let F be an endofunctor on $\mathcal{C}$. Then $\mathrm{Coalg}(\mathcal{B}, F)$ is a reflective replete subcategory of $\mathrm{Coalg}(\mathcal{C}, F)$.*

Proof. 1. The inclusion of $\mathcal{B}$ in $\mathcal{C}$ is monadic [2], so it preserves and creates limits.
2. Straightforward.

Examples of reflective replete subcategories:

– **Poset** of **Preord**, and **DiscSetoid** of **Setoid**. In each case the reflection is given by Q with unit p.
– **Setoid** of **Preord**. At A, the reflection is $(A_0, \equiv)$, where $\equiv$ is the least equivalence relation containing $\leqslant_A$, with unit id_{A_0}.

3 Relators

3.1 Relators and Simulation

Any notion of simulation depends on a way of transforming a relation. For example, given a relation $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\ } Y$, we define

– $\mathcal{P}X \xrightarrow{\mathrm{Sim}\mathcal{R}} \mathcal{P}Y$ to relate u to v when $\forall x \in u. \exists y \in v.\ x\ \mathcal{R}\ y$
– $\mathcal{P}X \xrightarrow{\mathrm{Bisim}\mathcal{R}} \mathcal{P}Y$ to relate u to v when $\forall x \in u. \exists y \in v.\ x\ \mathcal{R}\ y$ and $\forall y \in v.\ \exists x \in u.\ x\ \mathcal{R}\ y$.

for simulation and bisimulation respectively. In general:

Definition 6. *Let F be an endofunctor on* **Set**. *An F-*relator *maps each relation $X \xrightarrow{\mathcal{R}} Y$ to a relation $FX \xrightarrow{\Gamma\mathcal{R}} FY$ in such a way that the following hold.*

– *For any relations $X \xrightarrow{\mathcal{R},\mathcal{S}} Y$, if $\mathcal{R} \subseteq \mathcal{S}$ then $\Gamma\mathcal{R} \subseteq \Gamma\mathcal{S}$.*
– *For any set X we have $(=_{FX}) \subseteq \Gamma(=_X)$*
– *For any relations $X \xrightarrow{\mathcal{R}} Y \xrightarrow{\mathcal{S}} Z$ we have $(\Gamma\mathcal{R});(\Gamma\mathcal{S}) \subseteq \Gamma(\mathcal{R};\mathcal{S})$*
– *For any functions $Z \xrightarrow{f} X$ and $W \xrightarrow{g} Y$, and any relation $X \xrightarrow{\mathcal{R}} Y$, we have $\Gamma(f, g)^{-1}\mathcal{R} = (Ff, Fg)^{-1}\Gamma\mathcal{R}$.*

An F-relator Γ is conversive *when $\Gamma(\mathcal{R}^{\mathsf{c}}) = (\Gamma\mathcal{R})^{\mathsf{c}}$ for every relation $X \xrightarrow{\mathcal{R}} Y$.*

For example: Sim is a $\mathcal{P}$-relator, and Bisim is a conversive $\mathcal{P}$-relator.

We can now give a general definition of simulation.

Definition 7. *Let F be an endofunctor on* **Set***, and let Γ be an F-relator. Let M and N be F-coalgebras.*

1. *A* Γ-simulation *from M to N is a relation $M^{\cdot} \xrightarrow{\mathcal{R}} N^{\cdot}$ such that $\mathcal{R} \subseteq (\zeta_M, \zeta_N)^{-1}\Gamma\mathcal{R}$.*
2. *The largest Γ-simulation is called* Γ-similarity *and written $\lesssim^{\Gamma}_{M,N}$.*
3. *M is* Γ-encompassed *by N, written $M \preccurlyeq^{\Gamma} N$, when for every $x \in M$ there is $y \in N$ such that $x \lesssim^{\Gamma}_{M,N} y$ and $y \lesssim^{\Gamma}_{N,M} x$.*

For example: a Sim-simulation is an ordinary simulation, and a Bisim-simulation is a bisimulation.

The basic properties of simulations are as follows.

Lemma 5. *Let F be an endofunctor on* **Set***, and Γ an F-relator.*

1. *Let M be an F-coalgebra. Then $M \xrightarrow{(=_{M^{\cdot}})} M$ is a Γ-simulation. Moreover $\lesssim^{\Gamma}_{M,M}$ is a preorder on $M^{\cdot}$—an equivalence relation if Γ is conversive.*
2. *Let M, N, P be F-coalgebras. If $M \xrightarrow{\mathcal{R}} N \xrightarrow{\mathcal{S}} P$ are Γ-simulations then so is $M \xrightarrow{\mathcal{R};\mathcal{S}} P$. Moreover $(\lesssim^{\Gamma}_{M,N});(\lesssim^{\Gamma}_{N,P}) \sqsubseteq (\lesssim^{\Gamma}_{M,P})$.*
3. *Let M and N be F-coalgebras, and let Γ be conversive. If $M \xrightarrow{\mathcal{R}} N$ is a simulation then so is $N \xrightarrow{\mathcal{R}^{\mathsf{c}}} M$. Moreover $(\lesssim^{\Gamma}_{M,N})^{\mathsf{c}} = (\lesssim^{\Gamma}_{N,M})$ and $\lesssim^{\Gamma}_{M,N}$ is difunctional.*
4. *Let $M \xrightarrow{f} N$ and $M' \xrightarrow{g} N'$ be F-coalgebra morphisms. If $N \xrightarrow{\mathcal{R}} N'$ is a Γ-simulation then so is $M \xrightarrow{(f,g)^{-1}\mathcal{R}} M'$. Moreover $(f,g)^{-1}(\lesssim^{\Gamma}_{N,N'}) = (\lesssim^{\Gamma}_{M,M'})$.*
5. *$\preccurlyeq^{\Gamma}$ is a preorder on the class of F-coalgebras.*
6. *Let $M \xrightarrow{f} N$ be an F-coalgebra morphism. Then x and $f(x)$ are mutually Γ-similar for all $x \in M^{\cdot}$. Hence $M \preccurlyeq N$, and if f is surjective then also $N \preccurlyeq M$.*

Proof. We prove these statements in a different order from the one in which they were stated.

– For part (1), to show $(=_{M^{\cdot}})$ is a simulation we reason

$$\begin{aligned}(=_{M^{\cdot}}) &\sqsubseteq (\zeta_M, \zeta_M)^{-1}(=_{FM^{\cdot}}) \\ &\sqsubseteq (\zeta_M, \zeta_M)^{-1}\Gamma(=_{M^{\cdot}})\end{aligned}$$

We deduce reflexivity of $\lesssim^{\Gamma}_{M,M}$.

– For part (2), to show $\mathcal{R};\mathcal{S}$ is a simulation we reason

$$\begin{aligned}\mathcal{R};\mathcal{S} &\sqsubseteq (\zeta_M,\zeta_N)^{-1}\Gamma\mathcal{R};(\zeta_N,\zeta_P)^{-1}\Gamma\mathcal{S}\\ &\sqsubseteq (\zeta_M,\zeta_P)^{-1}(\Gamma\mathcal{R};\Gamma\mathcal{S})\\ &\sqsubseteq (\zeta_M,\zeta_P)^{-1}\Gamma(\mathcal{R};\mathcal{S})\end{aligned}$$

and the rest follows. We deduce the transitivity of $\lesssim^{\Gamma}_{M,M}$ in part (1).

– Part (5) is immediate.

– For part (3), to show $\mathcal{R}^{\mathsf{c}}$ is a simulation we reason

$$\begin{aligned}\mathcal{R}^{\mathsf{c}} &\subseteq ((\zeta_M,\zeta_N)^{-1}\Gamma\mathcal{R})^{\mathsf{c}}\\ &= (\zeta_N,\zeta_M)^{-1}((\Gamma\mathcal{R})^{\mathsf{c}})\\ &= (\zeta_N,\zeta_M)^{-1}\Gamma(\mathcal{R}^{\mathsf{c}})\end{aligned}$$

We deduce $(\lesssim^{\Gamma}_{M,N})^{\mathsf{c}} = (\lesssim^{\Gamma}_{N,M})$, and in part (1) we deduce symmetry of $\lesssim^{\Gamma}_{M,M}$. For difunctionality of $\lesssim^{\Gamma}_{M,N}$ we reason

$$\begin{aligned}(\lesssim^{\Gamma}_{M,N});(\lesssim^{\Gamma}_{M,N})^{\mathsf{c}};(\lesssim^{\Gamma}_{M,N}) &= (\lesssim^{\Gamma}_{M,N});(\lesssim^{\Gamma}_{N,M});(\lesssim^{\Gamma}_{M,N})\\ &\subseteq \lesssim^{\Gamma}_{M,N}\end{aligned}$$

– For part (4), to show $(f,g)^{-1}\mathcal{R}$ is a simulation, we reason

$$\begin{aligned}(f,g)^{-1}\mathcal{R} &\sqsubseteq (f,g)^{-1}(\zeta_M,\zeta_N)^{-1}\Gamma\mathcal{R}\\ &= (\zeta_{M'},\zeta_{N'})^{-1}(Ff,Fg)^{-1}\Gamma\mathcal{R} \quad (f,g \text{ coalgebra morphisms})\\ &= (\zeta_{M'},\zeta_{N'})^{-1}\Gamma(f,g)^{-1}\mathcal{R}\end{aligned}$$

We deduce $(f,g)^{-1}(\lesssim^{\Gamma}_{N,N'}) \subseteq (\lesssim^{\Gamma}_{M,M'})$.

– To prove part (6), we reason

$$\begin{aligned}(=_{M^{\cdot}}) &\sqsubseteq (f,f)^{-1}(=_{N^{\cdot}})\\ &= (X,f)^{-1}(f,Y)^{-1}(=_{N^{\cdot}})\\ &\sqsubseteq (M^{\cdot},f)^{-1}(f,N^{\cdot})^{-1}(\lesssim^{\Gamma}_{N,N})\\ &\sqsubseteq (M^{\cdot},f)^{-1}(\lesssim^{\Gamma}_{M,N})\end{aligned}$$

and likewise $(=_{M^{\cdot}}) \sqsubseteq (f,M^{\cdot})^{-1}(\lesssim^{\Gamma}_{N,M})$.

– To complete the proof of part (4) we reason

$$\begin{aligned}(\lesssim^{\Gamma}_{M,M'}) &= (=_{M^{\cdot}});(\lesssim^{\Gamma}_{M,M'});(=_{M'^{\cdot}})\\ &\sqsubseteq (f,M^{\cdot})^{-1}(\lesssim^{\Gamma}_{N,M});(\lesssim^{\Gamma}_{M,M'});(M'^{\cdot},g)^{-1}(\lesssim^{\Gamma}_{M',N'})\\ &= (f,g)^{-1}((\lesssim^{\Gamma}_{N,M});(\lesssim^{\Gamma}_{M,M'});(\lesssim^{\Gamma}_{M',N'}))\\ &\sqsubseteq (f,g)^{-1}(\lesssim^{\Gamma}_{N,N'})\end{aligned}$$

An F-coalgebra is *all-Γ-encompassing* when it is greatest in the $\preccurlyeq^{\Gamma}$ preorder. For example, take the disjoint union of all transition systems carried by an initial segment of $\mathbb{N}$. This is an all-Bisim-encompassing $\mathcal{P}^{[0,\aleph_0]}$-coalgebra, because every node of a $\mathcal{P}^{[0,\aleph_0]}$-coalgebra has only countably many descendants.

3.2 Relators Preserving Binary Composition

Definition 8. *Let F be an endofunctor on* **Set**. *An F-relator Γ is said to* preserve binary composition *when for all sets X, Y, Z and relations* $X \overset{\mathcal{R}}{\nrightarrow} Y \overset{\mathcal{S}}{\nrightarrow} Z$ *we have $\Gamma(\mathcal{R};\mathcal{S}) = (\Gamma\mathcal{R});(\Gamma\mathcal{S})$. If we also have $\Gamma(=_X) = (=_{FX})$ for every set X, then F is* functorial.

For example, Sim preserves binary composition and Bisim is functorial. We shall examine relators preserving binary composition using the following notions.

Definition 9.

1. *A commutative square* $\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[h]{} & W \end{array}$ *in* **Set** *is a* quasi-pullback *when*

$$\forall x \in X.\ \forall y \in Y.\ h(x) = k(y) \Rightarrow \exists z \in Z.\ x = f(z) \wedge g(z) = y$$

2. *A commutative square* $\begin{array}{ccc} C & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ A & \xrightarrow[h]{} & D \end{array}$ *in* **Preord** *is a* preorder-quasi-pullback *when* $\forall x \in A.\ \forall y \in B.\ h(x) \leqslant_D k(y) \Rightarrow \exists z \in C.\ x \leqslant_A f(z) \wedge g(z) \leqslant_B y$

Definition 10. *(adapted from [14]) Let F be an endofunctor on* **Set**. *A* stable preorder *on F is a functor $G : \mathbf{Set} \longrightarrow \mathbf{Preord}$ that makes*

$$\begin{array}{ccc} & & \mathbf{Preord} \\ & \overset{G}{\nearrow} & \downarrow{\scriptstyle (-)_0} \\ \mathbf{Set} & \xrightarrow[F]{} & \mathbf{Set} \end{array}$$

commute and sends quasi-pullbacks to preorder-quasi-pullbacks. It is a stable equivalence relation *on F when it is a functor $\mathbf{Set} \longrightarrow \mathbf{Setoid}$.*

For any relation $X \overset{\mathcal{R}}{\nrightarrow} Y$, we write $X \xleftarrow{\pi_{\mathcal{R}}} \mathcal{R} \xrightarrow{\pi'_{\mathcal{R}}} Y$ for the two projections. We can now give our main result.

Theorem 1. *Let F be an endofunctor on* **Set**. *There is a bijection between*

- *F-relators preserving binary composition*
- *stable preorders on F*

described as follows.

- *Given an F-relator Γ preserving binary composition, we define the stable preorder $\tilde{\Gamma}$ on F to map X to $(FX, \Gamma(=_X))$ and $X \xrightarrow{f} Y$ to Ff.*

– *Given a stable preorder G on F, we define the F-relator $\hat{G}$ to map a relation $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y$ to*

$$\{(x,y) \in FX \times FY \mid \exists z \in F\mathcal{R}.\ x \leqslant_{GX} (F\pi_{\mathcal{R}})z \wedge (F\pi'_{\mathcal{R}})z \leqslant_{GY} y\}$$

It restricts to a bijection between

– *conversive F-relators preserving binary composition*
– *stable equivalence relations on F.*

Proof. Let Γ be an F-relator preserving binary composition.

– Clearly $\tilde{\Gamma}X$ is a preordered set for any set X, and a setoid if Γ is conversive.
– Let $X \xrightarrow{f} Y$ be a function. Then

$$\begin{aligned}(=_X) &\subseteq (f,f)^{-1}(=_Y)\\ \Gamma(=_X) &\subseteq \Gamma(f,f)^{-1}(=_Y)\\ &= (Ff,Ff)^{-1}\Gamma(=_Y)\end{aligned}$$

so $\tilde{\Gamma}X \xrightarrow{Ff} \tilde{\Gamma}Y$ is monotone.
– Let $\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle k} \\ X & \xrightarrow[h]{} & W \end{array}$ be a quasi-pullback. Then

$$\begin{aligned}(h,k)^{-1}(=_W) &= (X,f)^{-1}(=_X);(g,Y)^{-1}(=_Y)\\ \therefore (Fh,Fk)^{-1}\Gamma(=_W) &= \Gamma(h,k)^{-1}(=_W)\\ &= \Gamma((X,f)^{-1}(=_X);(g,Y)^{-1}(=_Y))\\ &= \Gamma(X,f)^{-1}(=_X);\Gamma(g,Y)^{-1}(=_Y)\\ &= (FX,Ff)^{-1}\Gamma(=_X);(Fg,fY)^{-1}\Gamma(=_Y)\end{aligned}$$

i.e. the square

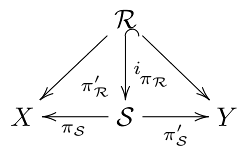
$$\begin{array}{ccc} \tilde{\Gamma}Z & \xrightarrow{Fg} & \tilde{\Gamma}Y \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Fk} \\ \tilde{\Gamma}X & \xrightarrow[Fh]{} & \tilde{\Gamma}W \end{array}$$

is a preorder-quasi-pullback.
– Let X and Y be sets and $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} Y$ a relation. Then

$$\begin{aligned}\mathcal{R} &= (X,\pi_{\mathcal{R}})^{-1}(=_X);(\pi'_{\mathcal{R}},Y)^{-1}(=_Y)\\ \therefore \Gamma\mathcal{R} &= \Gamma((X,\pi_{\mathcal{R}})^{-1}(=_X);(\pi'_{\mathcal{R}},Y)^{-1}(=_Y))\\ &= \Gamma(X,\pi_{\mathcal{R}})^{-1}(=_X);\Gamma(\pi'_{\mathcal{R}},Y)^{-1}(=_Y)\\ &= (FX,F\pi_{\mathcal{R}})^{-1}\Gamma(=_X);(F\pi'_{\mathcal{R}},FY)^{-1}\Gamma(=_Y)\\ &= \hat{\tilde{\Gamma}}\mathcal{R}\end{aligned}$$

We conclude that $\tilde{\Gamma}$ is a stable preorder on F—a stable equivalence relation if Γ is conversive—and $\Gamma = \hat{\tilde{\Gamma}}$. Conversely, suppose G is a stable preorder on F.

– Let X and Y be sets and $X \xrightarrow{\mathcal{R},\mathcal{S}} Y$ relations such that $\mathcal{R} \subseteq \mathcal{S}$. We have



where i is the inclusion of $\mathcal{R}$ in $\mathcal{S}$. For $x \in FX, y \in FY$, we have

$$\begin{aligned}(x,y) \in \hat{G}\mathcal{R} &\Leftrightarrow \exists z \in F\mathcal{R}.\ x \leqslant_{GX} (F\pi_{\mathcal{R}})z \wedge (F\pi'_{\mathcal{R}})z \leqslant_{GY} y \\ &\Leftrightarrow \exists z \in F\mathcal{R}.\ x \leqslant_{GX} (F\pi_{\mathcal{S}})(Fi)z \wedge (F\pi'_{\mathcal{R}})(Fi)z \leqslant_{GY} y \\ &\Rightarrow \exists w \in F\mathcal{S}.\ x \leqslant_{GX} (F\pi_{\mathcal{S}})w \wedge (F\pi'_{\mathcal{R}})w \leqslant_{GY} y \\ &\Leftrightarrow (x,y) \in \hat{G}\mathcal{S}\end{aligned}$$

giving $\hat{G}\mathcal{R} \subseteq \hat{G}\mathcal{S}$.

– Let X be a set. Both $\pi_{(=_X)}$ and $\pi'_{(=_X)}$ are inverse to the function $X \xrightarrow{\delta} (=_X)$ mapping $x \mapsto (x,x)$. For $x, x' \in FX$ we have

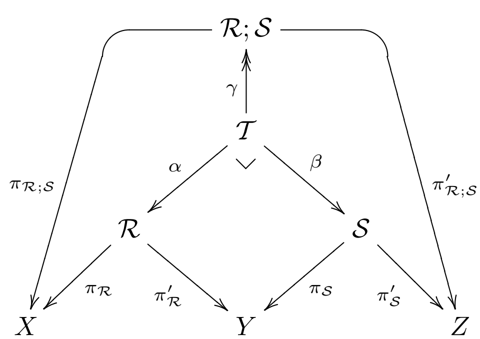
$$\begin{aligned}(x,x') \in \hat{G}(=_X) &\Leftrightarrow \exists z \in F(=_X).\ x \leqslant_{GX} (F\pi_{(=_X)})z \wedge F(\pi'_{(=_X)})z \leqslant_{GX} y \\ &\Leftrightarrow \exists x'' \in X.\ x \leqslant_{GX} x'' \wedge x'' \leqslant_{GX} x' \\ &\Leftrightarrow x \leqslant_{GX} x'\end{aligned}$$

giving $\hat{G}(=_X) = (\leqslant_{GX})$. We deduce both $(=_{FX}) \subseteq \hat{G}(=_X)$ and $\tilde{\hat{G}}X = GX$.

– Let X, Y, Z be sets and let $X \xrightarrow{\mathcal{R}} Y \xrightarrow{\mathcal{S}} Z$ be relations. Let

$$\mathcal{T} \stackrel{\text{def}}{=} \{(x,y,z) \mid (x,y) \in \mathcal{R} \wedge (y,z) \in \mathcal{S}\}$$

We have a diagram



where

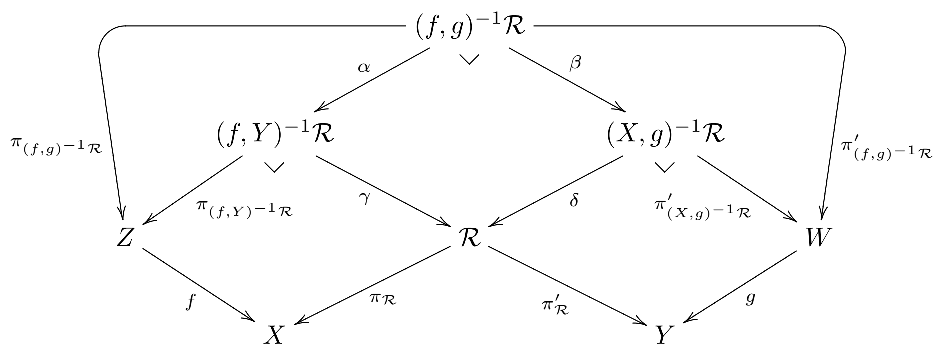
$$\alpha : (x, y, z) \mapsto (x, y)$$
$$\beta : (x, y, z) \mapsto (y, z)$$
$$\gamma : (x, y, z) \mapsto (x, z)$$

The $\lrcorner$ symbol indicates a pullback square and $\twoheadrightarrow$ a surjection. For any $x \in FX, z \in FZ$, we have

$$
\begin{aligned}
&(x, z) \in \hat{G}\mathcal{R}; \hat{G}\mathcal{S} \\
\Leftrightarrow\ & \exists y \in FY.\ (x, y) \in \hat{G}\mathcal{R} \wedge (y, z) \in \hat{G}\mathcal{S} \\
\Leftrightarrow\ & \exists y \in FY.\ \exists u \in F\mathcal{R}.\ \exists v \in F\mathcal{S}. \\
&\quad x \leqslant_{GX} (F\pi_{\mathcal{R}})u \wedge (F\pi'_{\mathcal{R}})u \leqslant_{GY} y \wedge y \leqslant_{GY} (F\pi_{\mathcal{S}})v \wedge (F\pi'_{\mathcal{S}})v \leqslant_{GZ} z \\
\Leftrightarrow\ & \exists u \in F\mathcal{R}.\ \exists v \in F\mathcal{S}.\ x \leqslant_{GX} (F\pi_{\mathcal{R}})u \wedge (F\pi'_{\mathcal{R}})u \leqslant_{GY} (F\pi_{\mathcal{S}})v \wedge (F\pi'_{\mathcal{S}})v \leqslant_{GZ} z \\
\Leftrightarrow\ & \exists u \in F\mathcal{R}.\ \exists v \in F\mathcal{S}. \\
&\quad x \leqslant_{GX} (F\pi_{\mathcal{R}})u \\
&\quad \wedge (\exists p \in F\mathcal{T}.\ u \leqslant_{G\mathcal{R}} (F\alpha)p \wedge (F\beta)p \leqslant_{G\mathcal{S}} v) \\
&\quad \wedge (F\pi'_{\mathcal{S}})v \leqslant_{GZ} z \qquad \text{(preorder-quasi-pullback property)} \\
\Leftrightarrow\ & \exists p \in F\mathcal{T}.\ x \leqslant_{GX} (F\pi_{\mathcal{R}})(F\alpha)p \wedge (F\pi'_{\mathcal{S}})(F\beta)p \leqslant_{GZ} z \\
&\qquad \text{(monotonicity of } F\pi_{\mathcal{R}} \text{ and } F\pi'_{\mathcal{S}}) \\
\Leftrightarrow\ & \exists p \in F\mathcal{T}.\ x \leqslant_{GX} (F\pi_{\mathcal{R};\mathcal{S}})(F\gamma)p \wedge (F\pi'_{\mathcal{R};\mathcal{S}})(F\gamma)p \leqslant_{GZ} z \\
\Leftrightarrow\ & \exists w \in F(\mathcal{R};\mathcal{S}). \\
&\quad x \leqslant_{GX} (F\pi_{\mathcal{R};\mathcal{S}})w \\
&\quad \wedge (\exists p \in F\mathcal{T}. w \leqslant_{G(\mathcal{R};\mathcal{S})} (F\gamma)p \wedge (F\gamma p) \leqslant_{G(\mathcal{R};\mathcal{S})} w) \\
&\quad \wedge (F\pi'_{\mathcal{R};\mathcal{S}})w \leqslant_{GZ} z \qquad \text{(monotonicity of } F\pi_{\mathcal{R};\mathcal{S}} \text{ and } F\pi'_{\mathcal{R};\mathcal{S}}) \\
\Leftrightarrow\ & \exists w \in F(\mathcal{R};\mathcal{S}).\ x \leqslant_{GX} (F\pi_{\mathcal{R};\mathcal{S}})w \wedge (F\pi'_{\mathcal{R};\mathcal{S}})w \leqslant_{GZ} z \\
&\qquad \text{(surjectivity up to preorder of } F\gamma) \\
\Leftrightarrow\ & (x, z) \in \hat{G}(\mathcal{R};\mathcal{S})
\end{aligned}
$$

giving $\hat{G}\mathcal{R}; \hat{G}\mathcal{S} = \hat{G}(\mathcal{R};\mathcal{S})$.

– Let $Z \xrightarrow{f} X$ and $W \xrightarrow{g} Y$ be functions and let $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;\;} Y$ be a relation. We have a diagram



where

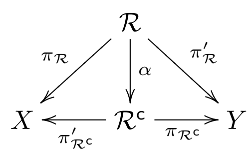
$$\begin{aligned}
\alpha &: (z,w) \mapsto (z, gw)\\
\beta &: (z,w) \mapsto (fz, w)\\
\gamma &: (z,y) \mapsto (fz, y)\\
\delta &: (x,w) \mapsto (x, gw)
\end{aligned}$$

For $z \in FZ, w \in FW$ we have

$$\begin{aligned}
&(z,w) \in (Ff,Fg)^{-1}\hat{G}\mathcal{R}\\
&\Leftrightarrow ((Ff)z,(Fg)w) \in \hat{G}\mathcal{R}\\
&\Leftrightarrow \exists t \in F\mathcal{R}.\ (Ff)z \leqslant_{GX} (F\pi_{\mathcal{R}})t \wedge (F\pi'_{\mathcal{R}})t \leqslant_{GY} (Fg)w\\
&\Leftrightarrow \exists t \in F\mathcal{R}.\\
&\qquad (\exists u \in F(f,Y)^{-1}\mathcal{R}.\ z \leqslant_{GX} (F\pi_{(f,Y)^{-1}\mathcal{R}})u \wedge (F\gamma)u \leqslant_{G\mathcal{R}} t)\\
&\qquad \wedge(\exists v \in F(X,g)^{-1}\mathcal{R}.\ t \leqslant_{G\mathcal{R}} (F\delta)v \wedge (F\pi'_{(X,g)^{-1}\mathcal{R}})v \leqslant_{GW} w)\\
&\hspace{20em}\text{(preorder-quasi-pullback property)}\\
&\Leftrightarrow \exists u \in F(f,Y)^{-1}\mathcal{R}.\ \exists v \in F(X,g)^{-1}\mathcal{R}.\\
&\qquad z \leqslant_{GX} (F\pi_{(f,Y)^{-1}\mathcal{R}})u \wedge (F\gamma)u \leqslant_{G\mathcal{R}} (F\delta)v \wedge (F\pi'_{(X,g)^{-1}\mathcal{R}})v \leqslant_{GW} w\\
&\Leftrightarrow \exists u \in F(f,Y)^{-1}\mathcal{R}.\ \exists v \in F(X,g)^{-1}\mathcal{R}.\\
&\qquad z \leqslant_{GX} (F\pi_{(f,Y)^{-1}\mathcal{R}})u\\
&\qquad \wedge(\exists p \in F(f,g)^{-1}\mathcal{R}.\ u \leqslant_{G(f,Y)^{-1}\mathcal{R}} (F\alpha)p \wedge (F\beta)p \leqslant_{G(f,W)^{-1}\mathcal{R}} v)\\
&\qquad \wedge(F\pi'_{(X,g)^{-1}\mathcal{R}})v \leqslant_{GW} w \qquad\qquad \text{(preorder-quasi-pullback property)}\\
&\Leftrightarrow \exists p \in F(f,g)^{-1}\mathcal{R}.\ z \leqslant_{GX} (F\pi_{(f,Y)^{-1}\mathcal{R}})(F\alpha)p \wedge (F\pi'_{(X,g)^{-1}\mathcal{R}})(F\beta)p \leqslant_{GW} w\\
&\hspace{14em}\text{(monotonicity of } F\pi_{(f,Y)^{-1}\mathcal{R}} \text{ and } F\pi'_{(X,g)^{-1}\mathcal{R}})\\
&\Leftrightarrow \exists p \in F(f,g)^{-1}\mathcal{R}.\ z \leqslant_{GZ} \pi_{(f,g)^{-1}\mathcal{R}}p \wedge \pi'_{(f,g)^{-1}\mathcal{R}}p \leqslant_{GW} w\\
&\Leftrightarrow (z,w) \in \hat{G}(f,g)^{-1}\mathcal{R}
\end{aligned}$$

giving $(Ff,Fg)^{-1}\hat{G}\mathcal{R} = \hat{G}(f,g)^{-1}\mathcal{R}$.

– Suppose that G is a stable equivalence relation on F, and let $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$ be a relation. Then we have a diagram



where the isomorphism $\alpha : (x, y) \mapsto (y, x)$. So for $x \in FX, y \in FY$ we have

$$\begin{aligned}
&(y, x) \in (\hat{G}\mathcal{R})^{\mathsf{c}} \\
&\Leftrightarrow (x, y) \in \hat{G}\mathcal{R} \\
&\Leftrightarrow \exists z \in F\mathcal{R}.\ x \leqslant_{GX} (F\pi_{\mathcal{R}})z \wedge (F\pi'_{\mathcal{R}})z \leqslant_{GY} y \\
&\Leftrightarrow \exists z \in F\mathcal{R}.\ y \leqslant_{GY} (F\pi'_{\mathcal{R}})z \wedge (F\pi_{\mathcal{R}}z) \leqslant_{GX} x \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(symmetry of } (\leqslant_{GX}) \text{ and } (\leqslant_{GY})) \\
&\Leftrightarrow \exists w \in F\mathcal{R}^{\mathsf{c}}.\ y \leqslant_{GY} (F\pi_{\mathcal{R}})z \wedge (F\pi'_{\mathcal{R}}z) \leqslant_{GX} x \\
&\Leftrightarrow (y, x) \in \hat{G}(\mathcal{R}^{\mathsf{c}})
\end{aligned}$$

giving $(\hat{G}\mathcal{R})^{\mathsf{c}} = \hat{G}(\mathcal{R}^{\mathsf{c}})$.

We conclude that $\hat{G}$ is an F-relator preserving binary composition, conversive if $\hat{G}$ is a stable equivalence relation, and that $\tilde{\hat{G}} = G$.

Corollary 1. *[4] Let F be an endofunctor on* **Set**.

1. *Suppose F preserves quasi-pullbacks. Then we obtain a conversive functorial F-relator $\hat{F}$ mapping a relation $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$ to*

$$\{(x, y) \in FX \times FY \mid \exists z \in F\mathcal{R}.\ x = (F\pi_{\mathcal{R}})z \wedge (F\pi'_{\mathcal{R}})z = y\}$$

2. *Let Γ be a functorial F-relator. Then F preserves quasi-pullbacks and $\Gamma = \hat{F}$.*

Proof. 1. ΔF is a stable equivalence relation on F. We also have

$$\widehat{\Delta F}(=_X) = (\leqslant_{\Delta FX}) = (=_{FX})$$

Therefore $\hat{F} = \hat{\Delta F}$ is a conversive functorial F-relator.

2. Since Γ is functorial, $\tilde{\Gamma} = \Delta F$. We deduce that ΔF maps quasi-pullbacks to order-quasi-pullbacks, i.e. that F preserves quasi-pullbacks; and also that $\Gamma = \hat{\Delta F} = \hat{F}$.

3.3 Further examples of relators

We first note several ways of constructing relators.

Lemma 6. *1. Let F be an endofunctor on* **Set***, and $(\Gamma_j)_{j\in J}$ a family of F-relators. Then*

$$\prod_{j\in J}\Gamma_j \;:\; (\;X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+}\; Y\;) \;\mapsto\; \bigcap_{j\in J}\Gamma_j\mathcal{R}$$

is an F-relator. If M and N are F-coalgebras, then $M^{\cdot} \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+} N^{\cdot}$ is a $\prod_{j\in J}\Gamma_j$-simulation from M to N iff, for all $j\in J$, it is a Γ_j-simulation from M to N.

2. Let F be an endofunctor on **Set***, and Γ an F-relator. Then*

$$\Gamma^{\mathsf{c}} \;:\; (\;X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+}\; Y\;) \;\mapsto\; (\Gamma\mathcal{R}^{\mathsf{c}})^{\mathsf{c}}$$

is an F-relator. If M and N are F-coalgebras, then $M^{\cdot} \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+} N^{\cdot}$ is a Γ^{c}-simulation from M to N iff $\mathcal{R}^{\mathsf{c}}$ is a Γ-simulation from N to M; hence $(\precsim^{\Gamma^{\mathsf{c}}}_{M,N}) = (\precsim^{\Gamma}_{N,M})^{\mathsf{c}}$.

3. Let F and G be endofunctors on **Set** *and $F \overset{\alpha}{\longrightarrow} G$ a natural transformation. Let Γ be an G-relator. Then*

$$\alpha^{-1}\Gamma \;:\; (\;X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+}\; Y\;) \;\mapsto\; (\alpha_X,\alpha_Y)^{-1}\Gamma\mathcal{R}$$

is an F-relator. If M and N are F-coalgebras, then $M^{\cdot} \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+} N^{\cdot}$ is an $\alpha^{-1}\Gamma$-simulation from M to N iff it is a Γ-simulation from $\mathrm{Coalg}(\mathbf{Set},\alpha)M$ to $\mathrm{Coalg}(\mathbf{Set},\alpha)N$; hence $(\precsim^{\alpha^{-1}\Gamma}_{M,N}) = (\precsim^{\Gamma}_{\mathrm{Coalg}(\mathbf{Set},\alpha)M,\mathrm{Coalg}(\mathbf{Set},\alpha)N})$.

4. The identity operation on relations is an $\mathsf{id}_{\mathbf{Set}}$-relator.

5. Let F and F' be endofunctors on **Set***. If Γ is an F-relator and Γ' an F'-relator, then $\Gamma'\Gamma$ is an $F'F$-relator.*

Proof. Trivial.

Note that $\Gamma \sqcap \Gamma^{\mathsf{c}}$ is the greatest conversive relator contained in Γ.

We give some relators for our examples:

- Via Def. 6(3), Sim and Bisim are $\mathcal{P}^{[0,\kappa)}$-relators and $\mathcal{P}^{[1,\kappa)}$-relators where κ is a cardinal or ∞. Moreover Sim preserves binary composition, and if $\kappa \leqslant 3$ or $\kappa \geqslant \aleph_0$ then Bisim is functorial. But for $4 \leqslant \kappa < \aleph_0$, the functors $\mathcal{P}^{[0,\kappa)}$ and $\mathcal{P}^{[1,\kappa)}$ do not preserve quasi-pullbacks, so Bisim does not preserve binary composition over them.

– We define $\mathcal{P}$Maybe-relators, all preserving binary composition. For a relation $X \xrightarrow{\mathcal{R}} Y$,

$$
\begin{aligned}
\text{LowerSim}\mathcal{R} &\stackrel{\text{def}}{=} \{(u,v) \in \mathcal{P}\text{Maybe}X \times \mathcal{P}\text{Maybe}Y \mid \\
&\quad \forall x \in \text{Just}^{-1}u.\ \exists y \in \text{Just}^{-1}v.\ (x,y) \in \mathcal{R}\} \\
\text{UpperSim}\mathcal{R} &\stackrel{\text{def}}{=} \{(u,v) \in \mathcal{P}\text{Maybe}X \times \mathcal{P}\text{Maybe}Y \mid \Uparrow \notin u \Rightarrow \\
&\quad \Uparrow \notin v \\
&\quad \wedge \forall y \in \text{Just}^{-1}v.\ \exists x \in \text{Just}^{-1}u.\ (x,y) \in \mathcal{R})\} \\
\text{ConvexSim} &\stackrel{\text{def}}{=} \text{LowerSim} \sqcap \text{UpperSim} \\
\text{SmashSim}\mathcal{R} &\stackrel{\text{def}}{=} \{(u,v) \in \mathcal{P}\text{Maybe}X \times \mathcal{P}\text{Maybe}Y \mid \Uparrow \notin u \Rightarrow \\
&\quad \Uparrow \notin v \\
&\quad \wedge \forall y \in \text{Just}^{-1}v.\ \exists x \in \text{Just}^{-1}u.\ (x,y) \in \mathcal{R} \\
&\quad \wedge \forall x \in \text{Just}^{-1}u.\ \exists y \in \text{Just}^{-1}v.\ (x,y) \in \mathcal{R}\} \\
\text{InclusionSim}\mathcal{R} &\stackrel{\text{def}}{=} \{(u,v) \in \mathcal{P}\text{Maybe}X \times \mathcal{P}\text{Maybe}Y \mid \\
&\quad \forall x \in \text{Just}^{-1}u.\ \exists y \in \text{Just}^{-1}v.\ (x,y) \in \mathcal{R}\} \\
&\quad \wedge \Uparrow \in u \Rightarrow \Uparrow \in v\}
\end{aligned}
$$

We respectively obtain notions of *lower*, *upper*, *convex*, *smash* and *inclusion simulation* on transiton systems with divergence [11,21]. By taking converses and intersections of these relators, we obtain—besides $\top$—nineteen different relators of which three are conversive. A more systematic analysis that includes these is presented in [17].

– We define D-relators. For a relation $X \xrightarrow{\mathcal{R}} Y$

$$
\begin{aligned}
\text{ProbSim}\mathcal{R} &\stackrel{\text{def}}{=} \{(d,d') \in DX \times DY \mid \forall U \subseteq X. dU \leqslant d'\mathcal{R}(U)\} \\
\text{ProbBisim}\mathcal{R} &\stackrel{\text{def}}{=} \{(d,d') \in DX \times DY \mid \forall U \subseteq X. dU \leqslant d'\mathcal{R}(U) \wedge d(\Uparrow) \leqslant d'(\Uparrow)\}
\end{aligned}
$$

where $\mathcal{R}(U) \stackrel{\text{def}}{=} \{y \in Y \mid \exists x \in U.\ (x,y) \in \mathcal{R}\}$, and see Lemma 7 below. We obtain notions of simulation and bisimulation on partial Markov chains as in [6,7,22,16,23]. By Thm. 1 of [15], ProbSim preserves binary composition and ProbBisim is functorial.

Lemma 7. ProbBisim *is the greatest conversive relator contained in* ProbSim.

Proof. [23] We first show it is conversive. Let $X \xrightarrow{\mathcal{R}} Y$ be a relation, and suppose that $(d,d') \in \text{ProbBisim}\mathcal{R}$. For any $V \subseteq Y$ we have $\mathcal{R}(X \setminus \mathcal{R}^{\mathsf{c}}(V)) \subseteq Y \setminus V$ giving

$$
\begin{aligned}
d'V &= 1 - d' \Uparrow - d'(Y \setminus V) \\
&\leqslant 1 - d' \Uparrow - d'\mathcal{R}(X \setminus \mathcal{R}^{\mathsf{c}}(V)) \\
&\leqslant 1 - d \Uparrow - d(X \setminus \mathcal{R}^{\mathsf{c}}(V)) \\
&= d\mathcal{R}^{\mathsf{c}}(V)
\end{aligned}
$$

$\mathcal{R}(X) \subseteq Y$ gives

$$\begin{aligned} d' \Uparrow &= 1 - d'Y \\ &\leqslant 1 - d'\mathcal{R}(X) \\ &\leqslant 1 - dX \\ &= d \Uparrow \end{aligned}$$

Thus $(d', d) \in \text{ProbBisim}\mathcal{R}^{\mathsf{c}}$ as required.

We therefore see that $(d, d') \in \text{ProbBisim}\mathcal{R}$ iff $(d, d') \in \text{Sim}\mathcal{R}$ and $(d', d) \in \text{Sim}\mathcal{R}^{\mathsf{c}}$. The result follows.

4 Theory of Simulation and Final Coalgebras

Throughout this section, F is an endofunctor on **Set** and Γ is an F-relator.

4.1 QF_Γ-coalgebras

Definition 11. *F_Γ is the endofunctor on* **Preord** *that maps A to $(FA_0, \Gamma(\leqslant_A))$ and $A \xrightarrow{f} B$ to Ff.*

Thus we obtain an endofunctor QF_Γ on **Preord**. It restricts to **Poset** and also, if Γ is conversive, to **Setoid** and to **DiscSetoid**.

For example, if A is a preordered set, then $Q\mathcal{P}^{[0,\aleph_0]}_{\text{Sim}}A$ is (isomorphic to) the set of countably generated lower sets, ordered by inclusion. The probabilistic case is unusual: D_{ProbSim} is already an endofunctor on **Poset**, so applying Q makes no difference (up to isomorphism). This reflects the fact that, for partial Markov chains, mutual similarity is bisimilarity [7].

A QF_Γ-coalgebra M is said to be *final* when the following equivalent conditions hold:

- M is final in Coalg(**Preord**, QF_Γ)
- M is final in Coalg(**Poset**, QF_Γ).

If Γ is conversive, the following are equivalent to the above:

- M is final in Coalg(**Setoid**, QF_Γ)
- M is final in Coalg(**DiscSetoid**, QF_Γ).

These equivalences follow from Lemma 4.

We adapt Def. 7 and Lemma 5 from F-coalgebras to QF_Γ-coalgebras.

Definition 12. *Let M and N be QF_Γ-coalgebras.*

1. *A* simulation *from M to N is a bimodule $M^{\cdot} \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} N^{\cdot}$ such that $\mathcal{R} \subseteq (\zeta_M, \zeta_N)^{-1}Q\Gamma\mathcal{R}$.*
2. *The greatest simulation is called* similarity *and written $\lesssim_{M,N}$.*

3. *M is* encompassed *by N, written $M \preccurlyeq N$, when for every $x \in M$ there is $y \in N$ such that $x \lesssim_{M,N} y$ and $y \lesssim_{N,M} x$.*

Lemma 8. *Let F be an endofunctor on* **Set**, *and Γ an F-relator.*

1. *Let M be a QF_Γ-coalgebra. Then $M \xrightarrow{(\leqslant_{M^\cdot})} M$ is a simulation. Moreover $\lesssim^\Gamma_{M,M}$ is a preorder on $M_0^\cdot$—an equivalence relation if Γ is conversive—that contains $\leqslant_{M^\cdot}$.*
2. *Let M, N, P be QF_Γ-coalgebras. If $M \xrightarrow{\mathcal{R}} N \xrightarrow{\mathcal{S}} P$ are simulations then so is $M \xrightarrow{\mathcal{R};\mathcal{S}} P$. Moreover $(\lesssim_{M,N});(\lesssim_{N,P}) \sqsubseteq (\lesssim_{M,P})$.*
3. *Let M and N be QF_Γ-coalgebras, and let Γ be conversive. If $M \xrightarrow{\mathcal{R}} N$ is a simulation then so is $N \xrightarrow{\overline{\mathcal{R}^{\mathsf{c}}}} M$ —recall that this is $(\leqslant_{N^\cdot});\mathcal{R}^{\mathsf{c}};(\leqslant_{M^\cdot})$. Moreover $(\lesssim_{M,N})^{\mathsf{c}} = (\lesssim_{N,M})$ and $\lesssim_{M,N}$ is difunctional.*
4. *Let $M \xrightarrow{f} N$ and $M' \xrightarrow{g} N'$ be QF_Γ-coalgebra morphisms. If $N \xrightarrow{\mathcal{R}} N'$ is a simulation then so is $M \xrightarrow{(f,g)^{-1}\mathcal{R}} M'$. Moreover $(\lesssim_{M,M'}) = (f,g)^{-1}(\lesssim_{N,N'})$.*
5. *$\preccurlyeq$ is a preorder on the class of QF_Γ-coalgebras.*
6. *Let $M \xrightarrow{f} N$ be an QF_Γ-coalgebra morphism. Then x and $f(x)$ are mutually similar for all $x \in M^\cdot$. Hence $M \preccurlyeq N$, and if f is surjective then also $N \preccurlyeq M$.*

Proof. We prove these statements in a different order from the one in which they were stated.

- For part (1), to show $\leqslant_{M^\cdot}$ is a simulation we reason

$$\begin{aligned}(\leqslant_{M^\cdot}) &\sqsubseteq (\zeta_M,\zeta_M)^{-1}(\leqslant_{QF_\Gamma M^\cdot}) && \text{(monotonicity of } \zeta)\\ &= (\zeta_M,\zeta_M)^{-1}(Q \leqslant_{F_\Gamma M^\cdot}) && \text{(Lemma 3(2))}\\ &= (\zeta_M,\zeta_M)^{-1}Q\Gamma(\leqslant_{M^\cdot})\end{aligned}$$

  We deduce that $\lesssim_{M,M}$ contains $\leqslant_{M^\cdot}$ and hence is reflexive.
- For part (2), to show $\mathcal{R};\mathcal{S}$ is a simulation we reason

$$\begin{aligned}\mathcal{R};\mathcal{S} &\sqsubseteq (\zeta_M,\zeta_N)^{-1}Q\Gamma\mathcal{R};(\zeta_N,\zeta_P)^{-1}Q\Gamma\mathcal{S}\\ &\sqsubseteq (\zeta_M,\zeta_P)^{-1}(Q\Gamma\mathcal{R};Q\Gamma\mathcal{S})\\ &= (\zeta_M,\zeta_P)^{-1}Q(\Gamma\mathcal{R};\Gamma\mathcal{S}) && \text{(by Lemma 3(3))}\\ &\sqsubseteq (\zeta_M,\zeta_P)^{-1}Q\Gamma(\mathcal{R};\mathcal{S})\end{aligned}$$

  and the rest follows. We deduce the transitivity of $\lesssim_{M,M}$ in part (1).
- Part (5) is immediate.

– For part (3), to show $\overline{\mathcal{R}^{\mathsf{c}}}$ is a simulation we reason

$$\begin{aligned}
\mathcal{R}^{\mathsf{c}} &\subseteq ((\zeta_M, \zeta_N)^{-1} Q\Gamma\mathcal{R})^{\mathsf{c}} \\
&= (\zeta_N, \zeta_M)^{-1}((Q\Gamma\mathcal{R})^{\mathsf{c}}) \\
&\sqsubseteq (\zeta_N, \zeta_M)^{-1}((\leqslant_{F_\Gamma N^{\cdot}}); (\Gamma\mathcal{R})^{\mathsf{c}}; (\leqslant_{F_\Gamma M^{\cdot}})) \quad \text{(by Lemma 3(5))} \\
&= (\zeta_N, \zeta_M)^{-1}((\leqslant_{F_\Gamma N^{\cdot}}); \Gamma(\mathcal{R}^{\mathsf{c}}); (\leqslant_{F_\Gamma M^{\cdot}})) \\
&= (\zeta_N, \zeta_M)^{-1}(\Gamma(\leqslant_{N^{\cdot}}); \Gamma(\mathcal{R}^{\mathsf{c}}); \Gamma(\leqslant_{M^{\cdot}})) \\
&= (\zeta_N, \zeta_M)^{-1}\Gamma((\leqslant_{N^{\cdot}}); \mathcal{R}^{\mathsf{c}}; (\leqslant_{N^{\cdot}}))
\end{aligned}$$

Since the RHS is a bimodule, it contains the bimodule closure of $\mathcal{R}^{\mathsf{c}}$, which must therefore be a simulation. We deduce $(\lesssim_{M,N})^{\mathsf{c}} = (\lesssim_{N,M})$, and in part (1) we deduce symmetry of $\lesssim_{M,M}$. For difunctionality of $\lesssim^{\Gamma}_{M,N}$ we reason

$$\begin{aligned}
(\lesssim_{M,N}); (\lesssim_{M,N})^{\mathsf{c}}; (\lesssim_{M,N}) &= (\lesssim_{M,N}); (\lesssim_{N,M}); (\lesssim_{M,N}) \\
&\subseteq \lesssim_{M,N}
\end{aligned}$$

– For part (4), to show $(f, g)^{-1}\mathcal{R}$ is a simulation, we reason

$$\begin{aligned}
(f, g)^{-1}\mathcal{R} &\sqsubseteq (f, g)^{-1}(\zeta_M, \zeta_N)^{-1} Q\Gamma\mathcal{R} \\
&= (\zeta_{M'}, \zeta_{N'})^{-1}(QF_\Gamma f, QF_\Gamma g)^{-1} Q\Gamma\mathcal{R} \quad (f, g \text{ coalgebra morphisms}) \\
&= (\zeta_{M'}, \zeta_{N'})^{-1} Q(F_\Gamma f, F_\Gamma g)^{-1}\Gamma\mathcal{R} \qquad \text{(by Lemma 3(4))} \\
&= (\zeta_{M'}, \zeta_{N'})^{-1} Q\Gamma(f, g)^{-1}\mathcal{R}
\end{aligned}$$

We deduce $(f, g)^{-1}(\lesssim_{N,N'}) \subseteq (\lesssim_{M,M'})$.

– To prove part (6), we reason

$$\begin{aligned}
(=_{M^{\cdot}}) &\sqsubseteq (f, f)^{-1}(=_{N^{\cdot}}) \\
&= (X, f)^{-1}(f, Y)^{-1}(=_{N^{\cdot}}) \\
&\sqsubseteq (M^{\cdot}, f)^{-1}(f, N^{\cdot})^{-1}(\lesssim_{N,N}) \\
&\sqsubseteq (M^{\cdot}, f)^{-1}(\lesssim_{M,N})
\end{aligned}$$

and likewise $(=_{M^{\cdot}}) \sqsubseteq (f, M^{\cdot})^{-1}(\lesssim_{N,M})$.

– To complete the proof of part (4) we reason

$$\begin{aligned}
(\lesssim_{M,M'}) &= (=_{M^{\cdot}}); (\lesssim_{M,M'}); (=_{M'^{\cdot}}) \\
&\sqsubseteq (f, M^{\cdot})^{-1}(\lesssim_{N,M}); (\lesssim_{M,M'}); (M'^{\cdot}, g)^{-1}(\lesssim_{M',N'}) \\
&= (f, g)^{-1}((\lesssim_{N,M}); (\lesssim_{M,M'}); (\lesssim_{M',N'})) \\
&\sqsubseteq (f, g)^{-1}(\lesssim_{N,N'})
\end{aligned}$$

We can also characterize coalgebra morphisms.

Lemma 9. *Let M and N be QF_Γ-coalgebras. For any function* $M_0^{\cdot} \xrightarrow{f} N_0^{\cdot}$*, the following are equivalent.*

1. $M \xrightarrow{f} N$ *is a* QF_Γ*-coalgebra morphism.*
2. $M \xrightarrow{(f,N_0^{\cdot})^{-1}(\leqslant_{N^{\cdot}})} N$ *and* $N \xrightarrow{(N_0^{\cdot},f)^{-1}(\leqslant_{N^{\cdot}})} M$ *are both simulations.*

Proof. ($\Rightarrow$) is immediate from Lemma 8. For ($\Leftarrow$), Lemma (1) tells us that $M^{\cdot} \xrightarrow{f} N^{\cdot}$ is monotone. We then observe

$$\begin{aligned}
(\leqslant_{M^{\cdot}}) &\sqsubseteq (f,f)^{-1}(\leqslant_{N^{\cdot}}) \\
&= (M^{\cdot},f)^{-1}(f,N^{\cdot})^{-1}(\leqslant_{N^{\cdot}}) \\
&\sqsubseteq (M^{\cdot},f)^{-1}\Psi_{M,N}(f,N^{\cdot})^{-1}(\leqslant_{N^{\cdot}}) \\
&= (M^{\cdot},f)^{-1}(\zeta_M,\zeta_N)^{-1}Q\Gamma(f,N^{\cdot})^{-1}(\leqslant_{N^{\cdot}}) \\
&= (\zeta_M,(f;\zeta_N))^{-1}Q(F_\Gamma f,F_\Gamma N^{\cdot})^{-1}\Gamma(\leqslant_{N^{\cdot}}) \\
&= (\zeta_M,(f;\zeta_N))^{-1}(QF_\Gamma f,QF_\Gamma N^{\cdot})^{-1}Q(\leqslant_{F_\Gamma N^{\cdot}}) \\
&= ((\zeta_M;QF_\Gamma f),(f;\zeta_N))^{-1}(\leqslant_{QF_\Gamma N^{\cdot}})
\end{aligned}$$

By the same argument $(\leqslant_{M^{\cdot}}) \sqsubseteq ((f;\zeta_N),(\zeta_M;QF_\Gamma f))^{-1}(\leqslant_{QF_\Gamma N^{\cdot}})$. By Lemma 2(1), since $QF_\Gamma N^{\cdot}$ is a poset, we have $f;\zeta_N = \zeta_M;QF_\Gamma f$ as required.

A QF_Γ-coalgebra N is *all-encompassing* when it is encompasses every $M \in \text{Coalg}(\mathbf{Preord}, QF_\Gamma)$, or equivalently every $M \in \text{Coalg}(\mathbf{Poset}, QF_\Gamma)$, or equivalently—if Γ is conversive—every $M \in \text{Coalg}(\mathbf{Setoid}, QF_\Gamma)$ or every $M \in \text{Coalg}(\mathbf{Setoid}, QF_\Gamma)$. These equivalences follow from the surjectivity of the units of the reflections.

4.2 Extensional Coalgebras

Definition 13. *An* extensional coalgebra *is* $M \in \text{Coalg}(\mathbf{Poset}, QF_\Gamma)$ *such that* $(\lesssim_{M,M}) = (\leqslant_{M^{\cdot}})$*. We write* ExtCoalg($\Gamma$) *for the category of extensional coalgebras and coalgebra morphisms.*

These coalgebras enjoy several properties.

Lemma 10. *Let* N *be an extensional coalgebra.*

1. *If* Γ *is conversive, then* $N^{\cdot}$ *is a discrete setoid.*
2. *Let* M *be a* QF_Γ*-coalgebra and* $N \xrightarrow{f} M$ *a coalgebra morphism. Then* f *is order-reflecting and injective.*
3. *Let* M *be a* QF_Γ*-coalgebra and* $M \xrightarrow{f} N$ *an order-reflecting, injective coalgebra morphism. Then* M *is extensional.*
4. *Let* M *be a* QF_Γ*-coalgebra such that* $M \preccurlyeq N$*. Then there is a unique* QF_Γ*-coalgebra morphism* $M \xrightarrow{f} N$ *.*

Proof. 1. Since $\lesssim_{N,N}$ has these properties.

2. It is an embedding because

$$\begin{aligned}(f,f)^{-1}(\leqslant_{M^{\cdot}}) &\sqsubseteq (f,f)^{-1}(\lesssim_{M,M}) \\ &= (\lesssim_{N,N}) \\ &= (\leqslant_{N^{\cdot}})\end{aligned}$$

and injective by Lemma 2(2).
3. $(\leqslant_{M^{\cdot}})$ is a poset by Lemma 2(3), and we then have

$$\begin{aligned}(\leqslant_{M^{\cdot}}) &= (f,f)^{-1}(\leqslant_{N^{\cdot}}) \\ &= (f,f)^{-1}(\lesssim_{N,N}) \\ &= (\lesssim_{M,M})\end{aligned}$$

4. For each $x \in M^{\cdot}$, define $f(x) \in N^{\cdot}$ to be the unique element such that $x \lesssim_{N,M} f(x)$ and $f(x) \lesssim_{M,N} x$. By Lemma 8(6) this is the only possibility for $f(x)$. Now for any $x \in M^{\cdot}$ and $y \in N^{\cdot}$ we have $x \lesssim_{M,N} y$ iff $f(x) \lesssim_{M,M} y$ i.e. iff $f(x) \leqslant_{N^{\cdot}} y$. So

$$\begin{aligned}(f, N^{\cdot}_0)^{-1}(\leqslant_{N^{\cdot}}) &= (\lesssim_{M,N}) \\ \text{Likewise } (N^{\cdot}_0, f)^{-1}(\leqslant_{N^{\cdot}}) &= (\lesssim_{N,M})\end{aligned}$$

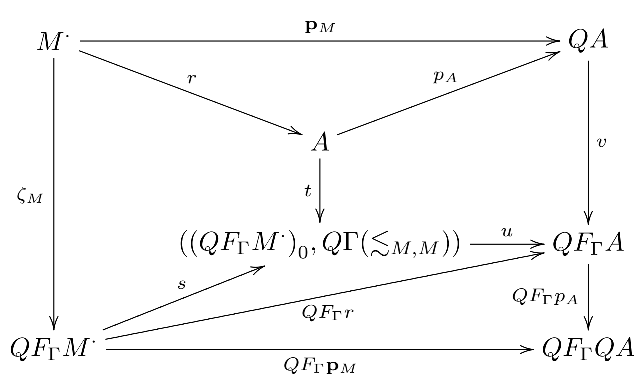
so Lemma 9 tells us that $M \xrightarrow{f} N$ is a QF_Γ-coalgebra morphism.

Thus ExtCoalg(Γ) is just a preordered class. It is a replete subcategory of Coalg($\mathbf{Poset}, QF_\Gamma$) and also—if Γ is conversive—of Coalg($\mathbf{DiscSetoid}, QF_\Gamma$). We next see that is reflective within Coalg($\mathbf{Preord}, QF_\Gamma$).

Lemma 11. *(Extensional Quotient) Let M be a QF_Γ-coalgebra, and define $\mathbf{p}_M \stackrel{\text{def}}{=} p_{(M^{\cdot}_0, \lesssim_{M,M})}$.*

1. *There is a QF_Γ-coalgebra $\mathbf{Q}M$ carried by $Q(M^{\cdot}_0, \lesssim_{M,M})$, uniquely characterized by the fact that $M \xrightarrow{\mathbf{p}_M} \mathbf{Q}M$ is a coalgebra morphism.*
2. *$\mathbf{Q}M$, with unit $\mathbf{p}_M$, is a reflection of M in* ExtCoalg(Γ).

Proof.
Put $A \stackrel{\text{def}}{=} (M^{\cdot}_0, \lesssim_{M,M})$. We then have a commutative diagram in **Preord**:



In this diagram,

- $M^{\cdot} \xrightarrow{r} A$ is given by $\mathsf{id}_{M^{\cdot}_0}$ and is monotone because $(\leqslant_{M^{\cdot}}) \sqsubseteq (\lesssim_{M,M})$.
- $QF_{\Gamma}M^{\cdot} \xrightarrow{s} ((QF_{\Gamma}M^{\cdot})_0, Q\Gamma(\lesssim_{M,M}))$ is given by $\mathsf{id}_{QF_{\Gamma}M^{\cdot}{}_0}$ and is monotone because

$$\begin{aligned}(\leqslant_{QF_{\Gamma}M^{\cdot}}) &= Q\Gamma(\leqslant_{M^{\cdot}}) \\ &\sqsubseteq Q\Gamma(\lesssim_{M,M})\end{aligned}$$

- $A \xrightarrow{t} ((QF_{\Gamma}M^{\cdot})_0, Q\Gamma(\lesssim_{M,M}))$ is given by ζ_M and is monotone because

$$\begin{aligned}(\leqslant_{A}) &= (\lesssim_{M,M}) \\ &\sqsubseteq (\zeta_M, \zeta_M)^{-1} Q\Gamma(\lesssim_{M,M})\end{aligned}$$

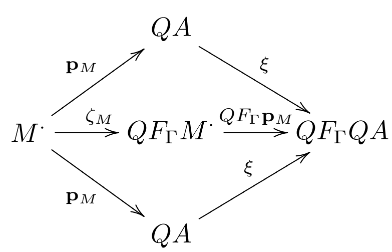
- $((QF_{\Gamma}M^{\cdot})_0, Q\Gamma(\lesssim_{M,M})) \xrightarrow{u} QF_{\Gamma}A$ is given by $QF_{\Gamma}r$ and is monotone because

$$\begin{aligned}Q\Gamma(\lesssim_{M,M}) &= Q\Gamma((r,r)^{-1}(\leqslant_{A})) \\ &= (QF_{\Gamma}r, QF_{\Gamma}r)^{-1} Q\Gamma(\leqslant_{A}) \\ &= (QF_{\Gamma}r, QF_{\Gamma}r)^{-1} (\leqslant_{QF_{\Gamma}A})\end{aligned}$$

- v is chosen, by the reflection property, to make the right-hand quadrilateral commute

All parts commute by the definition of the morphisms. We accordingly set $\mathbf{Q}M \stackrel{\text{def}}{=} (QA, (v; QF_{\Gamma}p_A))$ and we see that $\mathbf{p}_M$ is a coalgebra morphism from M to $\mathbf{Q}M$.

To show uniqueness, suppose (A, ξ) and (A, ξ') be two such coalgebras. Then



is a commutative diagram in **Preord**. Epicity of $\mathbf{Q}M$ gives $\xi = \xi'$.
Both $(\leqslant_{QA})$ and $(\lesssim_{\mathbf{Q}M,\mathbf{Q}M})$ are endobimodules on QA that are mapped by $(p_A, p_A)^{-1}$ to $\lesssim_{M,M}$. So by Lemma 3(1) they must be equal. Therefore $\mathbf{Q}M$ is extensional, and surjectivity of $\mathbf{p}_M$ gives $\mathbf{Q}M \preccurlyeq M$. Given another coalgebra morphism $M \xrightarrow{f} N$ with N extensional, we have $M \preccurlyeq N$ and hence $\mathbf{Q}M \preccurlyeq N$. So by Lemma 10(4) there is a unique coalgebra morphism $\mathbf{Q}M \xrightarrow{g} N$, and moreover $f = \mathbf{p}_M; g$.

More generally, a QF_Γ-coalgebra M can be quotiented by any $(\leqslant_{M^\cdot})$-containing preorder that is an endosimulation on M; but we shall not need this.

Lemma 12. *Let M be a QF_Γ-coalgebra. The following are equivalent.*

1. *M is a final QF_Γ-coalgebra.*
2. *M is all-encompassing and extensional.*
3. *M is extensional, and encompasses all extensional QF_Γ-coalgebras.*

Proof. (3) says that M is a final object in ExtCoalg(Γ), and this is equivalent to (1) by Lemma 4. (2) clearly implies (3), and is implied by the conjunction of (1) and (3).

Lemma 13. *Let M be a QF_Γ-coalgebra. The following are equivalent.*

1. *M is all-encompassing.*
2. *M encompasses all extensional coalgebras.*
3. *$\mathbf{Q}M$ is a final QF_Γ-coalgebra.*

Proof. Since the coalgebra morphism from M to $\mathbf{Q}M$ is surjective, these two coalgebras encompass each other.

(1) $\Rightarrow$ (2) Trivial.
(2) $\Rightarrow$ (3) $\mathbf{Q}M$ encompasses M, so it encompasses any extensional coalegbra, and it is extensional.
(3) $\Rightarrow$ (1) M encompasses $\mathbf{Q}M$ which by finality encompasses any QF_Γ-coalgebra.

4.3 Relating F-coalgebras and QF_Γ-coalgebras

We have studied F-coalgebras and QF_Γ-coalgebras separately, but now we connect them: each F-coalgebra gives rise to a QF_Γ-coalgebra, and the converse is also true in a certain sense.

Definition 14. *The functor $\Delta^\Gamma : \mathrm{Coalg}(\mathbf{Set}, F) \longrightarrow \mathrm{Coalg}(\mathbf{Preord}, QF_\Gamma)$ maps*

- *an F-coalgebra $M = (M^\cdot, \zeta_M)$ to the QF_Γ-coalgebra with carrier $\Delta M^\cdot$ and structure $\Delta M^\cdot \xrightarrow{\zeta_M} F_\Gamma \Delta M^\cdot \xrightarrow{p_{F_\Gamma \Delta M^\cdot}} QF_\Gamma \Delta M^\cdot$*
- *an F-coalgebra morphism $M \xrightarrow{f} N$ to f.*

Lemma 14. *Let M and N be F-coalgebras. Then a Γ-simulation from M to N is precisely a simulation from $\Delta^\Gamma M$ to $\Delta^\Gamma N$. Hence $(\lesssim_{\Delta^\Gamma M, \Delta^\Gamma N}) = (\lesssim^\Gamma_{M,N})$, and $M \preccurlyeq^\Gamma N$ iff $\Delta^\Gamma M \preccurlyeq \Delta^\Gamma N$.*

Proof. For any relation $M^\cdot \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!+\;\;} N^\cdot$ we have

$$\begin{aligned}
&((\zeta_M ; p_{F_\Gamma \Delta M^\cdot}), (\zeta_N ; p_{F_\Gamma \Delta N^\cdot}))^{-1} Q\Gamma\mathcal{R} \\
&\qquad = (\zeta_M, \zeta_N)^{-1} (p_{F_\Gamma \Delta^I M^\cdot}, p_{F_\Gamma \Delta N^\cdot})^{-1} Q\Gamma\mathcal{R} \\
&\qquad = (\zeta_M, \zeta_N)^{-1} \Gamma\mathcal{R}
\end{aligned}$$

The results follow immediately.

We are thus able to use a final QF_Γ-coalgebra to characterize similarity in F-coalgebras.

Theorem 2. *Let M be a final QF_Γ-coalgebra; for any QF_Γ-coalgebra P we write $P \xrightarrow{a_P} M$ for its anamorphism. Let N and N' be F-coalgebras. Then*

$$(\precsim^\Gamma_{N,N'}) = (a_{\Delta^\Gamma N}, a_{\Delta^\Gamma N'})^{-1}(\leqslant_{M^\cdot})$$

Proof. We have

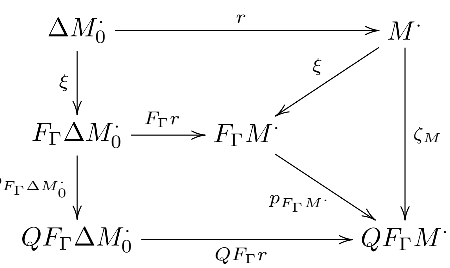
$$\begin{aligned}
(\precsim^\Gamma_{N,N'}) &= (\precsim_{\Delta^\Gamma N, \Delta^\Gamma N'}) && \text{(by Lemma 14)} \\
&= (a_{\Delta^\Gamma N}, a_{\Delta^\Gamma N'})^{-1}(\precsim_{M,M}) && \text{(by Lemma 8(4))} \\
&= (a_{\Delta^\Gamma N}, a_{\Delta^\Gamma N'})^{-1}(\leqslant_{M^\cdot}) && \text{(by extensionality of } M)
\end{aligned}$$

Our other results require moving from a QF_Γ-coalgebra to an F-coalgebra.

Lemma 15. *Let M be a QF_Γ-coalgebra. Then there is an F-coalgebra N and a surjective QF_Γ-coalgebra morphism $\Delta^\Gamma N \xrightarrow{f} M$.*

Proof. Using the Axiom of Choice, for each $x \in A$, choose $\xi(x) \in F_\Gamma A$ such that $\zeta_M(x) = [\xi(x)]_{F_\Gamma M^\cdot}$.

We thus obtain the following commutative diagram in **Preord**:



where $\Delta M_0^\cdot \xrightarrow{r} M^\cdot$ is given by $\mathsf{id}_{M_0^\cdot}$. The commutativity of the right hand triangle is by definition of ξ, and $M^\cdot \xrightarrow{\xi} F_\Gamma M^\cdot$ is monotone since

$$\begin{aligned}
(\leqslant_{M^\cdot}) &\sqsubseteq (\zeta_M, \zeta_M)^{-1}(\leqslant_{QF_\Gamma M^\cdot}) \\
&= (\zeta_M, \zeta_M)^{-1} Q(\leqslant_{F_\Gamma M^\cdot}) \\
&= (\xi, \xi)^{-1}(p_{F_\Gamma M^\cdot}, p_{F_\Gamma M^\cdot})^{-1} Q(\leqslant_{F_\Gamma M^\cdot}) \\
&= (\xi, \xi)^{-1}(\leqslant_{F_\Gamma M^\cdot})
\end{aligned}$$

The left-hand composite is $\Delta^\Gamma N$ so we are done.

Theorem 3.

1. *Let M be an F-coalgebra. Then $\mathbf{Q}\Delta^\Gamma M$ is a final QF_Γ-coalgebra iff M is all-Γ-encompassing.*

2. Any final QF_Γ-coalgebra is isomorphic to one of this form.

Proof. 1. By Lemma 13, $\mathbf{Q}\Delta^\Gamma M$ is final iff $\Delta^\Gamma M$ is all-encompassing. For $(\Rightarrow)$, given an F-coalgebra N, we know that $\Delta^\Gamma N \preccurlyeq \Delta^\Gamma M$ so by Lemma 14 $N \preccurlyeq M$. For $(\Leftarrow)$, given a QF_Γ-coalgebra N, Lemma 15 gives an F-coalgebra M' and surjective QF_Γ-coalgebra morphism $\Delta^\Gamma M' \xrightarrow{f} N$, so $N \preccurlyeq \Delta^\Gamma M'$. We know $M' \preccurlyeq^\Gamma M$, so Lemma 14 tells us that $M' \preccurlyeq^\Gamma M$ so $N \preccurlyeq^\Gamma M$.

2. Let N be a final QF_Γ-coalgebra. Lemma 15 gives us an F-coalgebra M and surjective coalgebra morphism $\Delta^\Gamma M \xrightarrow{f} N$, so $N \preccurlyeq \Delta^\Gamma M$. Since N is all-encompassing, $\Delta^\Gamma N$ is too. By Lemma 13, $\mathbf{Q}\Delta^\Gamma N$ is a final QF_Γ-coalgebra and hence isomorphic to N.

4.4 Coalgebras on Presheaves and Sheaves

Note This section is not used in the sequel.

Throughout our paper, F is an endofunctor on **Set**. However, we would like the results to hold if instead F is an endofunctor on a presheaf category, or even a sheaf category. However, we see that the proof of Lemma 15 (and therefore indirectly that of Thm. 3) uses the Axiom of Choice. Thus it adapts from **Set** to $\mathbf{Set}^S$ (where S is a set), but not to general presheaf or sheaf categories. Happily, under a mild assumption, we can prove Lemma 15 in these more general settings. We use the following concept.

A monotone function $A \xrightarrow{f} B$ is *dense* when for all $y \in B$ there is $x \in A$ such that $f(x) \leqslant_B y$ and $y \leqslant_B f(x)$. Clearly

- the composite of dense maps is dense
- if $f;g$ is dense then g is dense
- if $f;g$ is dense and g is an embedding then f is dense
- if f is surjective, it is dense
- if $A \xrightarrow{f} B$ is dense and B is a poset, then f is surjective.

Lemma 16. *Let $A \xrightarrow{f} B$ be a monotone function. Then f is dense iff Qf is dense.*

Proof. The following commutes in **Preord**:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_B} \\ QA & \xrightarrow[Qf]{} & QB \end{array} \qquad (1)$$

Suppose f is dense. Since p_B is surjective, the composite (1) is dense, so Qf is dense.

Conversely, suppose Qf is dense. Since p_A is surjective, the composite (1) is dense, and p_B is an embedding so f is dense.

In the following, $\mathbb{I}$ is a site, F is an endofunctor on $\mathbf{Sheaves}(\mathbb{I})$ and Γ is an F-relator.

Lemma 17. *The following are equivalent.*

- *The composite* $\mathbf{Set} \xrightarrow{\Delta} \mathbf{Preord} \xrightarrow{F_\Gamma} \mathbf{Preord}$ *sends surjections to dense functions.*
- F_Γ *sends dense functions to dense functions.*

Proof. ($\Leftarrow$) is trivial because Δ sends surjections to dense functions. For ($\Rightarrow$), let $A \xrightarrow{f} B$ be dense. The following commutes in $\mathbf{Preord}$:

$$\begin{array}{ccccc} F_\Gamma \Delta A_0 & \xrightarrow{Ff} & F_\Gamma \Delta B_0 & \xrightarrow{Fp_B} & F_\Gamma \Delta (QB)_0 \\ {\scriptstyle \mathsf{id}_{FA_0}}\downarrow & & & & \downarrow{\scriptstyle \mathsf{id}_{(QB)_0}} \\ F_\Gamma A & \xrightarrow[Ff]{} & F_\Gamma B & \xrightarrow[Fp_B]{} & F_\Gamma QB \end{array} \tag{2}$$

Now $A \xrightarrow{f;p_B} QB$ is dense hence surjective, so $A_0 \xrightarrow{f;p_B} (QB)_0$ is surjective, so $\Delta A_0 \xrightarrow{Ff;Fp_B} \Delta(QB)_0$ is dense. Also $\Delta(QB)_0 \xrightarrow{\mathsf{id}_{(QB)_0}} QB$ is surjective hence dense, so the composite (2) is dense. Hence $A \xrightarrow{Ff;Fp_B} QB$ is dense. Moreover $B \xrightarrow{p_B} QB$ is an embedding, so $F_\Gamma B \xrightarrow{Fp_B} F_\Gamma QB$ is an embedding, so $F_\Gamma A \xrightarrow{Ff} F_\Gamma B$ is dense.

When these conditions are satisfied, we say that Γ *respects density.* This is automatic if F preserves surjectivity (which must be the case if $\mathfrak{I}$ is a discrete category). If $F \xrightarrow{\alpha} G$ is a surjective natural transformation, and Γ is a G-relator respecting density, then the F-relator $F_{\alpha^{-1}\Gamma}$ also respects density.

The analogue of Lemma 15 is as follows.

Lemma 18. *Suppose that* Γ *respects density.*

Let M be a QF_Γ-coalgebra. Then there is an F-coalgebra N and a surjective QF_Γ-coalgebra morphism $\Delta^\Gamma N \xrightarrow{f} M$ *.*

Proof. We have a reflection $S \dashv U : \mathbf{Sheaves}(\mathbb{I}) \longrightarrow \mathbf{Set}^{\mathsf{ob}\,\mathbb{I}}$, where U is the forgetful functor, with unit η and surjective counit ε.

We write β for the counit of $\Delta \dashv (-)_0 : \mathbf{Preord}(\mathbf{Sheaves}(\mathbb{I})) \longrightarrow \mathbf{Sheaves}(\mathbb{I})$, so $\Delta A_0 \xrightarrow{\beta A} A$ is just id_{A_0}.

We put $A \stackrel{\text{def}}{=} F_\Gamma \Delta SUM^{\cdot}_0$ so $A_0 = FSUM^{\cdot}_0$, and let α be the composite

$$A \xrightarrow{p_A} QA \xrightarrow{QF_\Gamma \Delta \varepsilon M^{\cdot}_0} QF_\Gamma \Delta M^{\cdot}_0 \xrightarrow{QF_\Gamma \beta M^{\cdot}} QF_\Gamma M^{\cdot}$$

which is dense, since F_Γ preserves density, and therefore surjective, since $QF_\Gamma M^{\cdot}$ is a poset. Thus, using the Axiom of Choice, for each $i \in \mathsf{ob}\,\mathfrak{I}$ and $x \in M^{\cdot}_0 i$ pick $(\sigma i)x \in M^{\cdot}_0 i$ such that $(\zeta_M i)x = \alpha((\sigma i)x)$. We thus have a commutative diagram in $\mathbf{Set}^{\mathsf{ob}\,\mathfrak{I}}$

$$\begin{array}{ccc} UM^{\cdot}_0 & & \\ \downarrow{\scriptstyle\sigma} & \searrow^{\zeta_M} & \\ UFSUM^{\cdot}_0 = UA^{\cdot}_0 & \xrightarrow[\alpha]{} & U(QF_\Gamma M^{\cdot})_0 \end{array}$$

By the reflection, there is a unique natural transformation $SUM^{\cdot}_0 \xrightarrow{\xi} FSUM^{\cdot}_0$ such that

$$\begin{array}{ccc} UM^{\cdot}_0 & \xrightarrow{\eta UM^{\cdot}_0} & USUM^{\cdot}_0 \\ & {\scriptstyle\sigma}\searrow & \downarrow{\scriptstyle\xi} \\ & & UFSUM^{\cdot}_0 \end{array}$$

commutes. We set N to be $(SUM^{\cdot}_0, \xi)$. We have a commutative diagram in $\mathbf{Sheaves}(\mathbb{I})$

$$\begin{array}{ccc} SUM^{\cdot}_0 & \xrightarrow{\varepsilon M^{\cdot}_0} & M^{\cdot}_0 \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\zeta_M} \\ FSUM^{\cdot}_0 = A^{\cdot}_0 & \xrightarrow[\alpha]{} & (QF_\Gamma M^{\cdot})_0 \end{array} \tag{3}$$

using the reflection: if we apply U to both sides and prefix with $\eta UM^{\cdot}_0$ we obtain

$$\begin{array}{ccccc} USUM^{\cdot}_0 & \xleftarrow{\eta UM^{\cdot}_0} & UM^{\cdot}_0 & \xrightarrow{\eta UM^{\cdot}_0} & USUM^{\cdot}_0 \\ \downarrow{\scriptstyle\xi} & {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\mathsf{id}} & \downarrow{\scriptstyle\varepsilon M^{\cdot}_0} \\ UFSUM^{\cdot}_0 = UA^{\cdot}_0 & \xrightarrow[\alpha]{} & U(QF_\Gamma M^{\cdot})_0 & \xleftarrow[\zeta_M]{} & UM^{\cdot}_0 \end{array}$$

Finally (3) gives us the diagram in $\mathbf{Preord}(\mathbf{Sheaves}(\mathbb{I}))$

$$\begin{array}{ccccc} \Delta SUM^{\cdot}_0 & \xrightarrow{\Delta\varepsilon M^{\cdot}_0} & \Delta M^{\cdot}_0 & \xrightarrow{\beta M^{\cdot}} & M^{\cdot} \\ \downarrow{\scriptstyle\xi} & & & & \\ F_\Gamma \Delta SUM^{\cdot}_0 = A & & \overset{\alpha}{\searrow} & & \downarrow{\scriptstyle\zeta_M} \\ \downarrow{\scriptstyle p_A} & & & & \\ QA & \xrightarrow[QF_\Gamma \Delta\varepsilon M^{\cdot}_0]{} & QF_\Gamma \Delta M^{\cdot}_0 & \xrightarrow[QF_\Gamma \beta M^{\cdot}]{} & QF_\Gamma M^{\cdot} \end{array}$$

which says that the surjection

$$\Delta SUM^{\cdot}_0 \xrightarrow{\Delta\varepsilon M^{\cdot}_0} \Delta M^{\cdot}_0 \xrightarrow{\beta M^{\cdot}} M^{\cdot}$$

is a coalgebra morphism from $\Delta^{\Gamma} N$ to M.

Therefore we can conclude Thm. 3 under the assumption that Γ respects density.

5 Beyond Similarity

5.1 Multiple Relations

We recall from [10] that a *2-nested simulation* from M to N (transition systems) is a simulation contained in the converse of similarity. Let us say that a *nested preordered set* is a set equipped with two preorders $\leqslant_{\mathsf{n}}$ (think 2-nested similarity) and $\leqslant_{\mathsf{o}}$ (think converse of similarity) such that $(\leqslant_{\mathsf{n}}) \subseteq (\leqslant_{\mathsf{o}})$ and $(\leqslant_{\mathsf{n}}) \subseteq (\geqslant_{\mathsf{o}})$. It is a *nested poset* when $\leqslant_{\mathsf{n}}$ is a partial order. By working with these instead of preordered sets and posets, we can obtain a characterization of 2-nested similarity as a final coalgebra.

We fix a set I. For our example of 2-nested simulation, it would be $\{\mathsf{n}, \mathsf{o}\}$.

Definition 15. *(I-relations)*

1. *For any sets X and Y, an* I-relation $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ *is an I-indexed family $(\mathcal{R}_i)_{i \in I}$ of relations from X to Y. We write* $\mathrm{Rel}_I(X, Y)$ *for the complete lattice of I-relations ordered pointwise.*
2. *Identity I-relations $(=_X)$ and composite I-relations $\mathcal{R};\mathcal{S}$ are defined pointwise, as are inverse image I-relations $(f, g)^{-1}\mathcal{R}$ for functions f and g.*

We then obtain analogues of Def. 2 and 3. In particular, an *I-preordered set* A is a set A_0 equipped with an I-indexed family of preorders $(\leqslant_{A,i})_{i \in I}$, and it is an *$I$-poset* when $\bigcap_{i \in I} (\leqslant_i)$ is a partial order. We thus obtain categories $\mathbf{Preord}_I$ and $\mathbf{Poset}_I$, whose morphisms are *monotone* functions, i.e. monotone in each component. Given an I-preordered set A, the *principal lower set* of $x \in A$ is $\{y \in A \mid \forall i \in I.\ y \leqslant_{A,i} x\}$. The *quotient I-poset* QA is $\{[x]_A \mid x \in A\}$ with ith preorder relating $[x]_A$ to $[y]_A$ iff $x \leqslant_{A,i} y$, and we write $A \xrightarrow{p_A} QA$ for the function $x \mapsto [x]_A$. Thus $\mathbf{Poset}_I$ is a reflective replete subcategory of $\mathbf{Preord}_I$.

Returning to our example, a nested preordered set is a $\{\mathsf{n}, \mathsf{o}\}$-preordered set, subject to some constraints that we ignore until Sect. 5.2.

For the rest of this section, let F be an endofunctor on $\mathbf{Set}$, and Λ an *F-relator I-matrix*, i.e. an $I \times I$-indexed family of F-relators $(\Lambda_{i,j})_{i,j \in I}$. This gives us an operation on I-relations as follows.

Definition 16. *For any I-relation $FX \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} FY$, we define the I-relation $FX \overset{\Lambda\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} FY$ as $(\bigcap_{j \in I} \Lambda_{i,j} \mathcal{R}_j)_{i \in I}$.*

For our example, we take the $\mathcal{P}$-relator $\{\mathsf{n}, \mathsf{o}\}$-matrix TwoSim

$$\begin{aligned} \mathrm{TwoSim}_{\mathsf{n},\mathsf{n}} &\stackrel{\text{def}}{=} \mathrm{Sim} & \mathrm{TwoSim}_{\mathsf{n},\mathsf{o}} &\stackrel{\text{def}}{=} \mathrm{Sim}^{\mathsf{c}} \\ \mathrm{TwoSim}_{\mathsf{o},\mathsf{n}} &\stackrel{\text{def}}{=} \top & \mathrm{TwoSim}_{\mathsf{o},\mathsf{o}} &\stackrel{\text{def}}{=} \mathrm{Sim}^{\mathsf{c}} \end{aligned}$$

We can see that the operation $\mathcal{R} \mapsto \Lambda\mathcal{R}$ has the same properties as a relator.

Lemma 19.

1. *For any I-relations* $X \xrightarrow{\mathcal{R},\mathcal{S}} Y$ *, if* $\mathcal{R} \sqsubseteq \mathcal{S}$ *then* $\Lambda\mathcal{R} \sqsubseteq \Lambda\mathcal{S}$*.*
2. *For any set* X *we have* $(=_{FX}) \sqsubseteq \Lambda(=_X)$
3. *For any I-relations* $X \xrightarrow{\mathcal{R}} Y \xrightarrow{\mathcal{S}} Z$ *we have* $(\Lambda\mathcal{R});(\Lambda\mathcal{S}) \sqsubseteq \Lambda(\mathcal{R};\mathcal{S})$
4. *For any functions* $X' \xrightarrow{f} X$ *and* $Y' \xrightarrow{g} Y$ *and any I-relation* $X \xrightarrow{\mathcal{R}} Y$ *, we have* $\Lambda(f,g)^{-1}\mathcal{R} = (Ff, Fg)^{-1}\Lambda\mathcal{R}$*.*

Proof. Trivial.

Note by the way that TwoSim as a $\mathcal{P}$-relator matrix does not preserve binary composition. Now we adapt Def. 7.

Definition 17. *Let M and N be F-coalgebras.*

1. *A* Λ-simulation *from M to N is an I-relation* $M^{\cdot} \xrightarrow{\mathcal{R}} N^{\cdot}$ *such that for all* $i, j \in I$ *we have* $\mathcal{R}_i \in (\zeta_M, \zeta_N)^{-1}\Lambda_{i,j}\mathcal{R}_j$*, or equivalently* $\mathcal{R} \sqsubseteq \Lambda(\zeta_M, \zeta_N)^{-1}\mathcal{R}$*.*
2. *The largest Λ-simulation is called* Λ-similarity *and written* $\lesssim^{\Lambda}_{M,N}$*.*
3. *N is said to* Λ-encompass *M when for every $x \in M$ there is $y \in N$ such that, for all $i \in I$, we have* $x\ (\lesssim^{\Gamma}_{M,N,i})\ y$ *and* $y\ (\lesssim^{\Gamma}_{N,M,i})\ x$*.*

In our example, the n-component of $\lesssim^{\mathrm{TwoSim}}_{M,N}$ is 2-nested similarity, and the o-component is the converse of similarity from N to M.

The rest of the theory in Sect. 4 goes through unchanged, using Lemma 19.

5.2 Constraints

We wish to consider not all I-preordered sets (for a suitable indexing set I) but only those that satisfy certain constraints. These constraints are of two kinds:

– a "positive constraint" is a pair (i, j) such that we require $(\leqslant_i) \subseteq (\leqslant_j)$
– a "negative constraint" is a pair (i, j) such that we require $(\leqslant_i) \subseteq (\geqslant_j)$.

Furthermore the set of constraints should be "deductively closed". For example, if $(\leqslant_i) \subseteq (\geqslant_j)$ and $(\leqslant_j) \subseteq (\geqslant_k)$ then $(\leqslant_i) \subseteq (\leqslant_k)$.

Definition 18. *A* constraint theory *on I is a pair* $\gamma = (\gamma^+, \gamma^-)$ *of relations on I such that γ^+ is a preorder and* $\gamma^+;\gamma^-;\gamma^+ \subseteq \gamma^-$ *and* $\gamma^-;\gamma^- \subseteq \gamma^+$*.*

For our example, let γ_{nest} be the constraint theory on $\{\mathsf{n}, \mathsf{o}\}$ given by

$$\gamma^+_{\mathrm{nest}} = \{(\mathsf{n},\mathsf{n}), (\mathsf{n},\mathsf{o}), (\mathsf{o},\mathsf{o})\} \quad \gamma^-_{\mathrm{nest}} = \{(\mathsf{n},\mathsf{o})\}$$

A constraint theory γ gives rise to two operations γ^{+L} and γ^{-L} on relations (where L stands for "lower adjoint"). They are best understood by seeing how they are used in the rest of Def. 19.

Definition 19. *Let γ be a constraint theory on I.*

1. *For an I-relation* $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ *, we define I-relations*

   – $X \xrightarrow{\gamma^{+L}\mathcal{R}} Y$ *as* $(\bigcup_{j \in I(j,i) \in \gamma^+} \mathcal{R}_j)_{i \in I}$

   – $Y \xrightarrow{\gamma^{-L}\mathcal{R}} X$ *as* $(\bigcup_{j \in I(j,i) \in \gamma^-} \mathcal{R}_j^{\mathsf{c}})_{i \in I}$.

2. *An I-endorelation* $X \overset{\mathcal{R}}{\longrightarrow} X$ *is* γ-symmetric *when*

   – *for all $(j,i) \in \gamma^+$ we have $\mathcal{R}_j \subseteq \mathcal{R}_i$, or equivalently $\gamma^{+L}\mathcal{R} \sqsubseteq \mathcal{R}$*
   – *for all $(j,i) \in \gamma^-$ we have $\mathcal{R}_j^{\mathsf{c}} \subseteq \mathcal{R}_i$, or equivalently $\gamma^{-L}\mathcal{R} \sqsubseteq \mathcal{R}$.*

3. *We write* $\mathbf{Preord}_\gamma$ *(*$\mathbf{Poset}_\gamma$*) for the category of γ-symmetric I-preordered sets (I-posets) and monotone functions.*

4. *An I-relation* $X \overset{\mathcal{R}}{\longrightarrow} Y$ *is* γ-difunctional *when*

   – *for all $(j,i) \in \gamma^+$ we have $\mathcal{R}_j \subseteq \mathcal{R}_i$, or equivalently $\gamma^{+L}\mathcal{R} \sqsubseteq \mathcal{R}$*
   – *for all $(j,i) \in \gamma^-$ we have $\mathcal{R}_i; \mathcal{R}_j^{\mathsf{c}}; \mathcal{R}_i \subseteq \mathcal{R}_i$, or equivalently $\mathcal{R}; \gamma^{-L}\mathcal{R}; \mathcal{R} \sqsubseteq \mathcal{R}$.*

For our example, $\mathbf{Preord}_{\gamma_{\text{nest}}}$ and $\mathbf{Poset}_{\gamma_{\text{nest}}}$ are the categories of nested preordered sets and nested posets respectively. In general, $\mathbf{Poset}_\gamma$ is a reflective replete subcategory of $\mathbf{Preord}_\gamma$ and $\mathbf{Preord}_\gamma$ of $\mathbf{Preord}_I$.

Now let F be an endofunctor and Λ an F-relator I-matrix.

Definition 20. *Let γ be a constraint theory on I. Then Λ is* γ-conversive *when*

$$\bigsqcap_{\substack{l \in I \\ (l,k) \in \gamma^+}} \Lambda_{j,l} \sqsubseteq \Lambda_{i,k} \quad \text{for all } (j,i) \in \gamma^+ \text{ and } k \in I$$
$$\bigsqcap_{\substack{l \in I \\ (l,k) \in \gamma^-}} \Lambda_{j,l}^{\mathsf{c}} \sqsubseteq \Lambda_{i,k} \quad \text{for all } (j,i) \in \gamma^- \text{ and } k \in I$$

For our example, it is clear that the matrix TwoSim is γ_{nest}-conversive.

Lemma 20. *Let γ be a constraint theory on I such that Λ is γ-conversive. For every I-relation* $X \overset{\mathcal{R}}{\longrightarrow} Y$ *we have $\gamma^{+L}\Lambda\mathcal{R} \sqsubseteq \Lambda\gamma^{+L}\mathcal{R}$ and $\gamma^{-L}\Lambda\mathcal{R} \sqsubseteq \Lambda\gamma^{-L}\mathcal{R}$.*

Proof. Let $i \in I$. For all $j \in I$ such that $(j,i) \in \gamma^-$ and all $k \in I$ we have

$$\begin{aligned}
(\Lambda\mathcal{R})^{\mathsf{c}}_j &= \bigcap_{l \in I} (\Lambda_{j,l}\mathcal{R}_l)^{\mathsf{c}} \\
&\subseteq \bigcap_{\substack{l \in I \\ (l,k) \in \gamma^-}} (\Lambda_{j,l}\mathcal{R}_l)^{\mathsf{c}} \\
&= \bigcap_{\substack{l \in I \\ (l,k) \in \gamma^-}} \Lambda^{\mathsf{c}}_{j,l}\mathcal{R}^{\mathsf{c}}_l \\
&\subseteq \bigcap_{\substack{l \in I \\ (l,k) \in \gamma^-}} \Lambda^{\mathsf{c}}_{j,l} \bigcup_{\substack{m \in I \\ (m,k) \in \gamma^-}} \mathcal{R}^{\mathsf{c}}_m \\
&= \Big(\prod_{\substack{l \in I \\ (l,k) \in \gamma^-}} \Lambda^{\mathsf{c}}_{j,l}\Big) \bigcup_{\substack{m \in I \\ (m,k) \in \gamma^-}} \mathcal{R}^{\mathsf{c}}_m \\
&\subseteq \Lambda_{i,k} \bigcup_{\substack{m \in I \\ (m,k) \in \gamma^-}} \mathcal{R}^{\mathsf{c}}_m \\
&= \Lambda_{i,k}(\gamma^{-L}\mathcal{R})_k
\end{aligned}$$

and so

$$\begin{aligned}
(\gamma^{-L}\Lambda\mathcal{R})_i &= \bigcup{}_{(j,i) \in \gamma^-}^{j \in I} (\Lambda\mathcal{R})^{\mathsf{c}}_j \\
&\subseteq \bigcap_{k \in I} \Lambda_{i,k}(\gamma^{-L}\mathcal{R})_k \\
&= (\Lambda\gamma^{-L}\mathcal{R})_i
\end{aligned}$$

We conclude that $\gamma^{-L}\Lambda\mathcal{R} \sqsubseteq \Lambda\gamma^{-L}\mathcal{R}$ and similarly prove $\gamma^{+L}\Lambda\mathcal{R} \sqsubseteq \Lambda\gamma^{+L}\mathcal{R}$.

5.3 The Lattice of Constraint Theories

Let I be a set. Clearly the set of constraint theories, ordered by inclusion, form a complete lattice.

Lemma 21. *Let γ and γ' be constraint theories on I such that $\gamma \subseteq \gamma'$.*

1. *An I-endorelation $X \overset{\mathcal{R}}{\twoheadrightarrow} X$ that is γ'-symmetric is γ-symmetric.*
2. *An I-relation $X \overset{\mathcal{R}}{\twoheadrightarrow} Y$ that is γ'-difunctional is γ-difunctional.*
3. *Let F be an endofunctor on* **Set**. *An F-relator I-matrix Λ that is γ'-conversive is γ-conversive.*

Proof. Trivial.

Lemma 22. *Let T be a set of constraint theories on I, with supremum $\bigwedge T$.*

1. *Let* $X \overset{\mathcal{R}}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ *be an* I*-endorelation that, for all* $\gamma \in T$*, is* γ*-symmetric. Then* $\mathcal{R}$ *is* $\bigvee T$*-symmetric.*
2. *Let* F *be an endofunctor on* **Set***, and let* Λ *be an* F*-relator* I*-matrix that, for all* $\gamma \in T$*, is* γ*-conversive. Then* Γ *is* $\bigvee T$*-conversive.*

Proof. 1. Put

$$\delta^+ \stackrel{\text{def}}{=} \{(j,i) \in I \times I \mid \mathcal{R}_j \subseteq \mathcal{R}_i\}$$
$$\delta^- \stackrel{\text{def}}{=} \{(j,i) \in I \times I \mid \mathcal{R}^{\mathsf{c}}_j \subseteq \mathcal{R}_i\}$$

Then δ is a constraint theory on I containing every $\gamma \in T$. Hence it contains $\bigwedge T$.

2. Put

$$\delta^+ \stackrel{\text{def}}{=} \{(j,i) \in I \times I \mid \forall k \in I. \prod_{\substack{l \in I \\ (l,k) \in (\bigwedge T)^+}} \Lambda_{j,l} \sqsubseteq \Lambda_{i,k}\}$$
$$\delta^- \stackrel{\text{def}}{=} \{(j,i) \in I \times I \mid \forall k \in I. \prod_{\substack{l \in I \\ (l,k) \in (\bigwedge T)^-}} \Lambda^{\mathsf{c}}_{j,l} \sqsubseteq \Lambda_{i,k}\}$$

We show that δ is a constraint theory containing every $\gamma \in T$. Hence it contains $\bigwedge T$.

- Let $\gamma \in T$. If $(j,i) \in \gamma^-$ then for all $k \in I$

$$\begin{aligned} \prod_{\substack{l \in I \\ (l,k) \in (\bigwedge T)^-}} \Lambda_{j,l} &\sqsubseteq \prod_{\substack{l \in I \\ (l,k) \in \gamma^-}} \Lambda_{j,l} \\ &\sqsubseteq \Lambda^{\mathsf{c}}_{i,k} \end{aligned}$$

  since Λ is γ conversive; and so $(j,i) \in \gamma^-$. We conclude $\gamma^- \subseteq \delta^-$, and likewise $\gamma^+ \subseteq \delta^+$.
- Let $i \in I$. Then for all $k \in I$

$$\prod_{\substack{l \in I \\ (l,k) \in (\bigwedge T)^+}} \Lambda_{i,l} \sqsubseteq \Lambda_{i,k}$$

  because $(k,k) \in (\bigwedge T)^+$, so $(i,i) \in \delta^+$. So δ^+ is reflexive.
- Suppose $(j,i) \in \delta^-$ and $(i,h) \in \delta^-$. If $k \in I$, then for all $m \in I$ such that $(m,k) \in (\bigwedge T)^+$ and $l \in I$ such that $(l,m) \in (\bigwedge T)^+$, we have $(l,k) \in (\bigwedge T)^+$, so

$$\begin{aligned} \prod_{\substack{l \in I \\ (l,k) \in (\bigwedge T)^-}} \Lambda_{j,l} &\sqsubseteq \prod_{\substack{m \in I \\ (m,k) \in (\bigwedge T)^-}} \prod_{\substack{l \in I \\ (l,m) \in (\bigwedge T)^-}} \Lambda_{j,l} \\ &\sqsubseteq \prod_{\substack{m \in I \\ (m,k) \in (\bigwedge T)^-}} \Lambda^{\mathsf{c}}_{i,m} \\ &\sqsubseteq \Lambda_{h,k} \end{aligned}$$

So $(j, h) \in \delta^+$.
- The other requirements are verified similarly.

5.4 Generalized Theory of Simulation and Final Coalgebras (Sketch)

All the results of Sect. 4, in particular Thms. 2–3, generalize to the setting of a set I with a constraint theory γ. We replace "conversive" by "γ-conversive".

In our nested simulation example, we thus obtain an endofunctor $\mathcal{P}^{[0,\aleph_0]}_{\text{TwoSim}}$ on $\mathbf{Preord}_{\gamma_{\text{nest}}}$ that maps a nested preordered set $A = (A_0, (\leqslant_{A,\mathsf{n}}), (\leqslant_{A,\mathsf{o}}))$ to $(\mathcal{P}^{[0,\aleph_0]} A_0, \text{Sim}(\leqslant_{A,\mathsf{n}}) \cap \text{Sim}^{\mathsf{c}}(\leqslant_{A,\mathsf{o}}), \text{Sim}^{\mathsf{c}}(\leqslant_{A,\mathsf{o}}))$. We conclude:

- (from Thm. 2) Given a final $Q\mathcal{P}^{[0,\aleph_0]}_{\text{TwoSim}}$-coalgebra M, we can use $(\leqslant_{M\cdot,\mathsf{n}})$ and $(\geqslant_{M\cdot,\mathsf{o}})$ to characterize 2-nested similarity and similarity, respectively, in countably branching transition systems.
- (from Thm. 3) Given a countably branching transition system that is all-Bisim-encompassing (and hence all-TwoSim-encompassing), we can quotient it by 2-nested similarity to obtain a final $Q\mathcal{P}^{[0,\aleph_0]}_{\text{TwoSim}}$-coalgebra.

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