Thunkable implies central

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It is established in [1, Proposition 2.20] that, for a strong monad on a cartesian category $\mathcal{C}$, any Kleisli map that is thunkable is also central.[1] This note shows that (as expected) this generalizes to the setting where $\mathcal{C}$ is merely monoidal.

Firstly let $\mathcal{C}$ be a monoidal category with a monad T and left strength $t_{A,B} \colon TA \otimes B \to T(A \otimes B)$.

For a T-algebra (P, θ) and map $h \colon A \otimes \Delta \to P$, we write $h^{\sharp\theta}$ for the left Kleisli extension, i.e. the following composite:

$$TA \times \Delta \xrightarrow{t_{A,\Delta}} T(A \times \Delta) \xrightarrow{Th} TP \xrightarrow{\theta} P$$

Proposition 1. *For a map $f \colon \Gamma \to TA$, the following are equivalent.*

(a) The map f is thunkable*, i.e. the diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & TA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \eta_{TA}} \\ TA & \xrightarrow[T\eta_A]{} & T^2A \end{array}$$

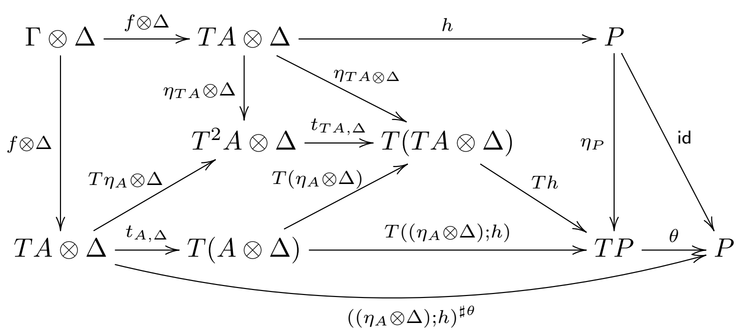
commutes.

(b) For any object Δ and T-algebra (P, θ) and map $h \colon \Delta \otimes TA \to P$, the diagram

$$\begin{array}{ccc} \Gamma \otimes \Delta & \xrightarrow{f\otimes\Delta} & TA \otimes \Delta \\ {\scriptstyle f\otimes\Delta}\downarrow & & \downarrow{\scriptstyle h} \\ TA \otimes \Delta & \xrightarrow[(\eta_A\otimes\Delta;h)^{\sharp\theta}]{} & P \end{array}$$

commutes.

Proof. For (a)⇒(b), we take



[1] Surprisingly, the converse is also true in the case of a continuation monad [4, Remark 3.5]. But in general a central map need not be thunkable, even if it is an isomorphism. For example, the writer monad $\mathbb{Z}_2 \times -$ on **Set** is commutative, so every Kleisli map is central, and in particular the Kleisli map $1 \to 1$ sending $* \mapsto (1, *)$ is a central involution that is not thunkable, cf. [3, Section 5.2].

For (b)⇒(a), we take Δ to be 1 and ignore $- \otimes 1$, and we take (P, θ) to be the free algebra on TA. Then we have

$$\begin{array}{ccccc}
\Gamma & & \xrightarrow{\quad f \quad} & & TA \\
{\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle \eta_{TA}} \\
TA & & \xrightarrow{(\eta_A;\eta_{TA})^{\sharp\theta}} & & \\
{\scriptstyle T\eta_A}\downarrow & & & & \\
T^2A & \xrightarrow{T\eta_{TA}} & T^3A & \xrightarrow{\mu_{TA}} & T^2A \\
& & \underset{\mathsf{id}}{\longrightarrow} & &
\end{array}$$

□

Corollary 1. *For any algebra (P, θ) and maps $h, k \colon TA \otimes \Delta \to P$, the following are equivalent.*

(a) The diagram

$$\begin{array}{ccc}
A \otimes \Delta & \xrightarrow{\eta_A \otimes \Delta} & TA \otimes \Delta \\
{\scriptstyle \eta_A \otimes \Delta}\downarrow & & \downarrow{\scriptstyle k} \\
TA \otimes \Delta & \xrightarrow[h]{} & P
\end{array}$$

commutes.

(b) For any object Γ and thunkable $f \colon \Gamma \to TA$, the diagram

$$\begin{array}{ccc}
\Gamma \otimes \Delta & \xrightarrow{f \otimes \Delta} & TA \otimes \Delta \\
{\scriptstyle f \otimes \Delta}\downarrow & & \downarrow{\scriptstyle k} \\
TA \otimes \Delta & \xrightarrow[h]{} & P
\end{array} \qquad (1)$$

commutes.

Proof. The implication (a)⇒(b) follows from Proposition 1(a)⇒(b). For (b)⇒(a), put $\Gamma = A$ and $f = \eta_A$. □

Now suppose that T has bistrength consisting of $t_{A,B} \colon TA \otimes B \to T(A \otimes B)$ and $t'_{A,B} \colon A \otimes TB \to T(A \otimes B)$. (Recall from [2] that "bistrength" means that the two maps $(A \otimes TB) \otimes C \longrightarrow T((A \otimes B) \otimes C)$ are always equal. While this condition is not used in our argument, it is needed to ensure that the Kleisli category is premonoidal, specifically that the associator $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ is natural in B. In the case of a symmetric monoidal category, it follows from the condition that t and t' correspond across the symmetry. I do not know whether there are interesting examples of bistrong monads other than these.)

For maps $f \colon A \to TB$ and $g \colon C \to TD$, the condition that f commutes with g is equivalent to the instance of (b) where (P, θ) is the free algebra on $A \otimes B$ and h is the composite

$$TA \otimes \Delta \xrightarrow{TA \otimes g} TA \otimes TB \xrightarrow{t_{A,TB}} T(A \otimes TB) \xrightarrow{Tt'_{A,B}} T^2(A \otimes B) \xrightarrow{\mu_{A \otimes B}} T(A \otimes B)$$

and k is the composite

$$TA \otimes \Delta \xrightarrow{TA \otimes g} TA \otimes TB \xrightarrow{t'_{TA,B}} T(TA \otimes B) \xrightarrow{Tt_{A,B}} T^2(A \otimes B) \xrightarrow{\mu_{A \otimes B}} T(A \otimes B)$$

So if f is thunkable then it commutes with g. So thunkability implies left centrality, and likewise it implies right centrality.

References

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